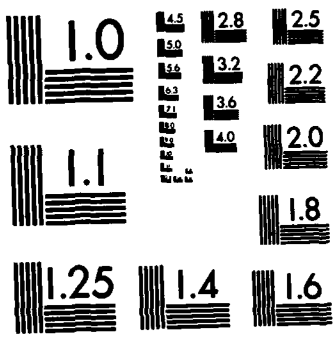
AD-A123 288 AN EXTENDED METHOD FOR THE ALLOCATION OF EXPLORATORY DEVELOPMENT RESOURCES IN LOGISTICS(U) SRI INTERNATIONAL MENLO PARK CA H A OLENDER OCT 82 N00167-82-K-0019 1/1

UNCLASSIFIED F/G 15/5 NL

END

FILMED

DTIC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

Final Report — *October 1982*

AN EXTENDED METHOD FOR THE ALLOCATION OF EXPLORATORY DEVELOPMENT RESOURCES IN LOGISTICS

By: HENRY A. OLENDER

Prepared for:

DAVID W. TAYLOR NAVAL SHIP RESEARCH
AND DEVELOPMENT CENTER
BETHESDA, MARYLAND 20084

CONTRACT N00167-82-K-0019

333 Ravenswood Avenue
Menlo Park, California 94025 U.S.A.
(415) 326-6200
Cable: SRI INTL MPK
TWX 910-373-2046

SRI

Final Report *October 1982*

AN EXTENDED METHOD FOR THE ALLOCATION OF EXPLORATORY DEVELOPMENT RESOURCES IN LOGISTICS

By: HENRY A. OLENDER

Prepared for:

DAVID W. TAYLOR NAVAL SHIP RESEARCH
AND DEVELOPMENT CENTER
BETHESDA, MARYLAND 20084

CONTRACT N00167-82-K-0019

SRI Project 3943

Approved for public release; distribution unlimited.

Approved by:

JACQUES NAAR, *Director*
Center for Defense Analysis

DAVID D. ELLIOTT, *Vice President*
Research and Analysis Division

333 Ravenswood Avenue • Menlo Park, California 94025 • U.S.A.
(415) 326-6200 • Cable: SRI INTL MPK • TWX: 910-373-2046

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER	2. GOVT ACCESSION NO. AD-A123288	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) AN EXTENDED METHOD FOR THE ALLOCATION OF EXPLORATORY DEVELOPMENT RESOURCES IN LOGISTICS		5. TYPE OF REPORT & PERIOD COVERED Final Report Covering the Period December 1981 through August 1982
		6. PERFORMING ORG. REPORT NUMBER SRI Project 3943
7. AUTHOR(s) Henry A. Olender		8. CONTRACT OR GRANT NUMBER(s) Contract N00167-82-K-0019
9. PERFORMING ORGANIZATION NAME AND ADDRESS SRI International 333 Ravenswood Avenue Menlo Park, California 94025		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS David W. Taylor Naval Ship Research and Development Center, Code 187 Bethesda, Maryland 20084		12. REPORT DATE October 1982 / 13. NO. OF PAGES 89
		15. SECURITY CLASS. (of this report) UNCLASSIFIED
14. MONITORING AGENCY NAME & ADDRESS (if diff. from Controlling Office)		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Resource allocation method
Attribute-value tradeoffs
Multi-attribute decisions
Preference function analysis
Logistics measures of effectiveness
Exploratory development measure of effectiveness

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This report describes further work done on the development of a resource allocation method for logistics exploratory development. This further work was concerned with extending previously developed methods that were based on local linear estunates of indifference functions to incorporate nonlinear effects. Several types of nonlinear effects were identified and analyzed to determine their implications on the basic method. A piecewise linear model was developed to estimate the indifference functions. Procedures for quantifying

19. KEY WORDS (Continued)

20 ABSTRACT (Continued)

the piecewise linear model were developed and described. A hypothetical numerical example was constructed and used to illustrate the key elements of the method.

The basic linear method which was extended in this work applies to the problem of selecting a preferred exploratory development program from among a number of alternative programs whose benefits are measured by a number of separate and disparate measures of effectiveness. The method assumes that each program can be funded for the same budget. The method is based on a series of subjective tradeoff assessments made by the decision maker. These tradeoff assessments involve only two of the measures of effectiveness at a time. The procedures then determine the implied preference of the decision maker between any pair of alternative programs through the consideration of a set of intermediate hypothetical programs described only in terms of their expected benefits.

CONTENTS

LIST OF ILLUSTRATIONS vii

LIST OF TABLES ix

PREFACE xi

I INTRODUCTION 1

A. Background 1

B. The Problem 3

1. Measures of Effectiveness 3
2. Resource Allocation for Exploratory Development 4
3. The Basic RA Method 4

C. Research Objectives and Approach 6

II SUMMARY AND CONCLUSIONS 7

A. Summary 7

B. Conclusions 9

III THE BASIC RA METHOD 11

A. Concept and Elements of the RA Method 11

B. Review of the Basic RA Method 12

IV NONLINEAR EFFECTS IN MOE TRADEOFFS 23

A. General 23

B. Categorization of Nonlinear Indifference Curves 24

1. Threshold 26
2. Convexity 27
3. Complementarity 28
4. Disutility 29
5. Proportionality 31

C. Implications of the Nonlinear Categories 31

1. Threshold 31
2. Proportionality 32
3. Disutility 32
4. Complementarity 33

V THE EXTENDED RA METHOD 35

A. Development of the Extended RA Method 35
B. Procedures of the Extended RA Method 44

1. Tradeoff Assessment Requirements for Step 6 46
2. Piecewise Linear Optimization Model for Step 7 51

VI ILLUSTRATIVE EXAMPLE RA PROBLEM 53

A. Problem Description 53
B. Tradeoff Assessment Information 56
C. Most Preferred Proportion Determination 58
D. Hyperplane Determinations 59
E. The Objective Function Formulation 61
F. Tradeoff Assessments Revisited (Consistency Checking) . . . 62

APPENDIX A A-1

A. Forming and Selecting Hyperplanes A-2
B. Tradeoff Ratio Bounds and Inferences A-7

APPENDIX B B-1
APPENDIX C C-1

ILLUSTRATIONS

III-1	Hypothetical Outcome Sequence	13
III-2	Tradeoff Assessment Tableau	16
III-3	Graphical Optimization	19
IV-1	Linearity Case	25
IV-2	Threshold Case	26
IV-3	Convexity Case	27
IV-4	Complementarity Case	29
IV-5	Disutility Case	30
V-1	Piecewise Linear Indifference Curves	35
V-2	Piecewise Linear Graphical Optimization	39
VI-1	Indifference Curves (Equipment vs. Spare Modules)	54
VI-2	Indifference Curves (Equipment vs. Maintenance Personnel)	54
VI-3	Indifference Curves (Spare Modules vs. Maintenance Personnel)	55
B-1	Piecewise Linear Approximation	B-2
B-2	Linear Functional Representations	B-3
C-1	Most Preferred Proportion Parameters	C-2

TABLES

V-I Maximum Number of Allowed Hyperplanes 43

V-2 Tradeoff Ratio Matrix 47

V-3 Table of Tradeoff Inferences 48

PREFACE

This report documents the analysis and findings of a research project conducted for the David W. Taylor Naval Ship Research and Development Center (DTNSRDC). Bethesda, Maryland. The sponsor and technical monitor was M.J. Zubkoff, Code 187, of DTNSRDC. The work was performed under Contract N00167-82-K-0019.

The research was performed in the Center for Defense Analysis (CDA) of the Research and Analysis Division (RAD) of SRI International. J. Naar is Director of CDA, and D.D. Elliott is Executive Director of RAD.

H.A. Olender was project leader and principal investigator. He was assisted by R.H. Monahan and L.C. Goheen.

I INTRODUCTION

A. Background

The David W. Taylor Naval Ship Research and De[illegible]opment Center (DTNSRDC) is the lead laboratory for naval logist[illegible]. In addition, the Technical Strategist for Logistics and Faci[illegible] is located there. It is the task of the Technical Strat[illegible] to develop and maintain an overall technical strategy to focus the thrust of all exploratory development (ED) in the field of naval logistics (certain specific logistic functions are assigned to other technical strategists). This approach to planning ED is innovative, especially for the area of naval logistics.

Naval logistics is heterogeneous, comprising a wide variety of very different and quite technical functions. These functions require different expertise, employ different technologies, and are evaluated by different measures of effectiveness. As a result, at the supporting establishment level (where most research and development is conducted), naval logistics has been largely planned, managed, and conducted in separate functional areas by separate agencies--e.g., Naval Supply System Command or Naval Sea Systems Command. For the most part, ED has been conducted according to the needs felt within each functional area with only broadbrush coordination among functional areas.

However, the Technical Strategist for Logistics is required to view naval logistics as a whole. He is to identify the regions of needed improvement, the pertinent emerging technologies to meet these needs, and the potential payoff in ED of technologies to

meet the needs. Then he must recommend, from the alternative combinations of separate functional area ED programs, the integrated program that will result in the greatest benefit to overall naval logistics system effectiveness for the budget available.

Much work remains to be done before the process of developing and maintaining a technical strategy for logistics is perfected. A pressing near-term requirement is a methodology for allocating ED resources among and within the key areas. A longer-term requirement is the development of a method to model the overall naval logistics system in order to measure the impact of changes in elements of the logistics system on fleet readiness or total system costs.

These two requirements are related. Proper allocation of the ED funds requires the knowledge of measures of effectiveness (MOEs) for the logistics system, and these MOEs are derived from the different steps required to model the overall naval logistics system.

Key tasks associated with a resource allocation method are developing meaningful and useful MOEs and establishing explicit or implicit relationships (where they exist) among the various MOEs to better understand their impact on overall effectiveness; and developing a method of ED resource allocation for trading off the expected achievable levels of the MOEs that characterize each program.

This research program is a continuation of previous research initiated at SRI for DTNSRDC. One of these projects resulted in the development of a general resource allocation (RA) method for selection of ED programs characterized by multiple disparate MOE outcomes. The resulting RA method is based on the subjective but informed judgment of a decision maker (DM) to provide MOE preference information.

In the initial development of the RA method, relatively simple preference structures among the naval logistics MOEs were assumed for local modelling of the DMs global preferences. This led to an RA procedure that can be employed iteratively to determine the most preferred ED program from among some set of programs. The existence of more complex preference structures and their impact of the basic RA procedures were recognized as areas requiring further investigation. Thus, the purpose of the present research is to characterize the more complex preference structures, and refine the RA method as appropriate.

B The Problem

1. Measures of Effectiveness

In each functional area of logistics, different MOEs have been defined to measure different aspects of performance--e.g., in the supply system, one MOE for operational performance is requisition fill rate, and for financial performance, one MOE is the ratio of sales to value of inventory. These are valid MOEs from the viewpoint of a supply officer at a supply depot. Different but related MOEs will be of concern to the user, such as an operational commander. He will be primarily concerned with the response time for the system to supply him with a certain type of part or quantity of material. This response time will be a function of, among other functional MOEs, the requisition fill rate mentioned above. Thus, no simple MOE is now, or may ever be, available to measure all important aspects of effectiveness for an entire functional area, and in some cases the MOEs used may be mutually conflicting.

Among different functional areas--e.g., the supply system and the maintenance system--the relationships between MOEs is even more ill defined. Finally, in the overall system the interplay among different functional MOEs and their cumulative effect on the evaluation of the overall system effectiveness are only poorly defined.

2. Resource Allocation for Exploratory Development

Currently the methods of arriving at the ED resource allocation decisions within the relatively short deadlines imposed by budget schedules depend mainly on judgment, experience, and intuition. Without a formal method for allocating ED resources among the heterogeneous key areas of logistics, the decisions are difficult to make and the rationale followed in the selection may be hard to reconstruct.

The difficulty lies in the fact that each key-area technology program is characterized by a set of expected achievable levels of different but important MOEs. Each of these characteristics or attributes measures a different type of effectiveness, and they cannot now be objectively and quantitatively traded off to determine the preferred program or the order of preference of other programs. A subjective methodology is required, which is based on judgmental inputs by decision-makers.

3. The Basic RA Method

In previous work, the Basic RA Method was developed to specifically address the question of how to compare alternatives whose expected outcomes are multifaceted.* It relies heavily on the subjective model relating the needs of the Navy to fulfill its mission, the various logistics MOEs that relate to the Navy's capability to carry out this mission, and the relative effects of improvements in these MOEs on this capability. The method allows the DM to progressively build up and communicate his preferences concerning specific ED programs and their expected outcomes expressed as achievable levels of important MOEs. He does this through a sequence of MOE tradeoff assessments between two alternatives that differ only in the values of two MOEs. These tradeoff assessments result in the construction of a sequence of hypothetical alternatives that link two real alternatives, and

*H.A. Olender, "A Method for the Allocation of Exploratory Development Resources in Logistics," SRI International Report 6549, December 1978.

allow the inference of a preference (or ranking) between these two alternatives. Applying this approach sequentially to all alternatives results in a relative ranking among them.

The key concept of the method is that the set of hypothetical alternatives serve as surrogate alternatives for one of the real alternatives in the sense that the DM is indifferent between obtaining the set of outcome MOEs for the real alternative and for any one of the hypothetical alternatives.

As tradeoff assessment information is expressed and accumulated, an analytical model of the DMs preferenced structure can be constructed and employed to rank remaining alternatives without further tradeoff assessments.

The analytical model employed can be characterized as a linear tradeoff model because it assumes that the tradeoff ratios among the MOEs are constant. This results in linear indifference functions. The validity and utility of the analytical model depends on the degree to which the linear assumption is approximated by the DM's true preference function.

Various types of tradeoff nonlinearities may exist among MOEs that may require modification or extension of some of the Basic RA Method procedures. For example, the utility and preference for certain combinations of MOE outcome values may depend on acquiring some minimum level of one or more MOE values. Other MOEs may be required in certain proportion for greatest utility. In some cases, the exceedance of a given MOE outcome level may cause a resulting disutility.

The basic tradeoff assessment procedures when applied to all candidate ED alternatives will generally identify the most preferred ED program regardless of the presence of the above nonlinear effects. However, the analytical model and procedures that are part of the Basic RA Method may perform poorly under these circumstances. Thus, one research issue is how to modify or extend the Basic RA Method to handle the nonlinear tradeoff effects.

C. Research Objectives and Approach

The objectives of the research reported here were to:

(1) Investigate the implications of nonlinear preference structures on the Basic RA Method (a linear indifference model) previously developed at SRI.

(2) Extend or revise the RA method to include procedures for incorporating nonlinear preference models into the methodology.

The approach consisted of three phases. The first phase consisted of an attempt to understand, more comprehensively, the types of nonlinear effects that may be encountered for the various naval logistics MOEs. The output of this phase basically consists of a characterization of MOE tradeoffs and identification of expected types of nonlinearities.

The second phase consisted of devising and examining various techniques for modelling the significant types of nonlinear effects identified in Phase 1. One of the approaches, a piecewise linear approach, was then selected as most appropriate for extending the basic linear model. The development of the mathematical basis and procedures for this model was then completed.

Finally, in the third phase, a numerical example was constructed and some of the procedures were applied to illustrate the method. This example is purely hypothetical and only illustrates the numerical part of the procedure after the appropriate tradeoff assessments have been performed.

II SUMMARY AND CONCLUSIONS

A. Summary

This research has considered a variety of types of nonlinear effects that may characterize the indifference relationship between pairs of MOEs. These nonlinearities have been categorized as:

1. Threshold Effects
2. Convexity
3. Complementarity
4. Disutility
5. Proportionality

We have focussed on methods of modelling the convexity and complementarity cases with a piecewise linear indifference model. The remaining three nonlinearity cases appear to have the primary implications of either eliminating alternative ED programs prior to the application of the RA method
s, or reducing the number of MOEs that must be processed.

Threshold effects, for example, result in the requirement for a given level of one or both MOEs before any significant utility is attached to the values of the MOEs. This type of nonlinearity suggests that alternative ED programs that have outcomes below the threshold be discarded prior to the application of the RA methods developed under this project. The disutility and proportionality effects on the other hand imply a strong relationship between the two MOEs that can be modelled by a functional relationship. Again, only alternatives with outcomes that satisfy this relationship should be evaluated by the RA methods described in this report. The strong relationship between these types of MOEs suggests that a single derived MOE can be used to replace the pair

of related MOEs. Alternately, either one or the other MOE can be retained for processing by the RA methods to serve as surrogate for the derived MOE.

The piecewise linear model developed in this research increases the number of tradeoff assessments required from the DM as we would expect. This nonlinear model which can be incorporated into the Basic RA Method to form what we have called the Extended RA Method requires tradeoff assessments for two distinct purposes. The first purpose is to define the indifference tradeoff ratios from which hyperplanes can be formed and incorporated into the model. These tradeoff assessments must include both negative and positive tradeoff assessments about some outcome point. The distinction between these two tradeoff assessments is the direction of movement from the reference outcome point and is either negative or positive depending on whether the DM's response is required to be in the negative or positive direction, respectively.

The second purpose of the tradeoff assessments is to define a most preferred marginal proportion direction for increasing MOE values. This information provides weighting factors that are associated with each hyperplane employed in the piecewise linear model.

The piecewise linear model includes the strictly linear model employed in the Basic RA Method. The number of tradeoff assessments of both types for an arbitrary n-dimensional problem cannot be specified ahead of time. At the minimum if the model turns out to be strictly linear, only 2(n-1) indifference tradeoff assessments will be required, and there will be no need to determine the most preferred marginal proportion direction. At the other extreme, n(n-1) indifference assessments will be required in addition to (n-1) tradeoff assessments for determining the most preferred marginal proportion direction.

The procedures for generating the piecewise linear model parameters are developed and described within this report. These include:

1) Indifference tradeoff assessments
2) Tradeoff ratio inferences
3) Tradeoff ratio consistency evaluation
4) Most preferred marginal proportion tradeoff assessments
5) Determination and selection of model hyperplanes
6) Objective function formation

Finally, an illustrative numerical example has been constructed and processed through these six steps to demonstrate the new procedures.

B. Conclusions

The Extended RA Method provides a method of modelling nonlinearities in indifference relationships that can be characterized as convexity or complementarity cases. As in the Basic RA Method, a sequence of pairwise tradeoff assessments between MOEs in required. Thus, an advantage of this model is that its interface with the DM remains the same as the previous linear model.

However, the cost of the more complex piecewise linear indifference model is a significant increase in the number of tradeoff assessments required from the DM. This not only increases his load directly, but also indirectly in terms of insuring a consistent set of tradeoff assessment responses. In this context, the desired consistency is with an increasing marginal rate of substitution model.

Consistency checking and correcting procedures have been included in the Extended RA Method procedures, as illustrated with the numerical example, and these should make the consistency problem more tractable.

It is difficult to judge at this point whether the utility of the Extended RA Method is worth the increased load on the DM. The answer to this issue will depend on the dimensionality of the nonlinearity relationships. When only one or two MOEs require a nonlinear model, the tradeoff assessment load is automatically reduced since many of the remaining tradeoff ratio values required can be inferred. It is our judgment that only in this latter case will the Extended RA Method be justified relative to the Basic RA Method.

III THE BASIC RA METHOD

A. Concept and Elements of the RA Method

The Basic RA Method was developed to deal with resource allocation problems characterized by alternative ED programs whose outcomes are measured by multiple MOEs (i.e., outcomes are vectors). One prime difficulty in assessing the relative worth of these ED programs is the need for simultaneous comparison of and tradeoff assessments among the different values of the different types of MOEs.

The essence of the Basic RA Method is to decompose the multiple MOE assessment problem into a sequence of simpler tradeoff assessment tasks. This is accomplished by selecting two alternative ED program expected outcomes and constructing a sequence of hypothetical outcomes that eventually link the real outcomes as described below. Each hypothetical outcome differs from the two sequentially adjacent hypothetical outcomes in only two MOEs. This is also true for the first real outcome and the first hypothetical outcome. Comparison of any adjacent pair of outcome vectors then involves only a "pairwise tradeoff assessement" between the two differing MOEs. The sequence is also constructed so that the last hypothetical outcome differs from the second real outcome in only one MOE.

The application of the pairwise tradeoff assessments works in such a manner as to construct a sequence of hypothetical outcomes such that the DM is indifferent among them, and indifferent between the first real outcome and any hypothetical outcome. Finally, since the last hypothetical outcome and the second real outcome differ in only one MOE, a direct preference assessment can be made between them based on the values of that singular MOE.

For example, if the MOE is "good" (i.e., more is preferred to less), the outcome with the higher MOE value is preferred. The preference relationship between the two real outcomes, and thus, the preference between the two real alternative ED programs, is established by induction.

> The key concept of the method is that the set of hypothetical alternatives serve as surrogate alternatives for one of the real alternatives in the sense that the DM is indifferent between obtaining the sets of outcome MOEs for the real alternative and for any one of the hypothetical alternatives.

In addition to the procedures outlined above, the accumulation of tradeoff assessment information can be employed to contruct a linear model that locally approximates the DM's indifference function. This model provides a real-valued objective function that can be employed in an optimization procedure to determine the most preferred alternative ED programs.

B. Review of the Basic RA Method

The construction of the sequence of hypothetical outcomes employed in the Basic RA Method is illustrated in Figure III-1. Assume that two real alternative ED programs with four MOEs have outcome vectors A and B given by

$$A = (a_1, a_2, a_3, a_4) \qquad (1)$$

and

$$B = (b_1, b_2, b_3, b_4) \quad . \tag{2}$$

$$\begin{array}{lcccccc}
 & & MOE_1 & MOE_2 & MOE_3 & MOE_4 & \\
A & = (& a_1 & a_2 & a_3 & a_4 &) \\
A' & = (& b_1 & c_2 & a_3 & a_4 &) \\
A'' & = (& b_1 & b_2 & c_3 & a_4 &) \\
A''' & = (& b_1 & b_2 & b_3 & c_4 &) \\
B & = (& b_1 & b_2 & b_3 & b_4 &)
\end{array}$$

FIGURE III-1 HYPOTHETICAL OUTCOME SEQUENCE

Three hypothetical outcomes, designated by A´, A´´, and A´´´ are also shown in Figure III-1. Note that the two attributes in which each successive pair of programs differ are indicated within each dashed box.

Comparing A' to A, we see that A' has the same MOE values except for the first pair of MOEs. The value of MOE_1 for A' is set equal to the value of MOE_1 for B, and the value of MOE_2 is set equal to c_2. The procedure for obtaining c_2 (and the other c_i's) is the pairwise tradeoff assessment and will be described shortly. In a similar manner A'' has the same MOE values as A' except for the pair of MOEs consisting of MOE_2 and MOE_3. Through this stepwise procedure we progress from hypothetical outcomes that more closely match A to hypothetical outcomes that more closely match B.

The hypothetical outcomes in Figure III-1 are obtained by the assignment of values to c_2, c_3, and c_4 by the DM. He chooses these values so that he will be indifferent between any adjacent pair of outcomes. Thus, the construction of the hypothetical outcomes does not entail lengthy analysis to establish the values of the MOEs nor does it imply that there exists a feasible ED program that can produce that outcome.

The pairwise tradeoff assessment is the procedure for eliciting the required information from the DM. He provides this information by responding to the following type of question. Given two outcomes whose MOE values differ in all but two MOEs, and assume that the level of one MOE for the first outcome can be increased to the level of the second outcome, how much can the level of the second MOE for the first outcome be decreased to make you indifferent between the new outcome and the first outcome? Stated another way, how much of one MOE are you willing to tradeoff for another? The quantitative response of the DM provides the value of the c's discussed above.

After the construction of A''', we see from Figure III-1 that the preference between A''' and B can be determined solely on the basis of the values for MOE_4. If b_4 is equal to c_4, the DM is indifferent between A''' and B; if b_4 exceeds c_4, he prefers A'''.

Since he is indifferent between A and A‴, his preference between A and B is established. In this process, we see that A‴ (as well as A′ and A″) serves as a surrogate for A.

The primary objective of constructing the hypothetical programs is to relieve the DM of the task of assessing simultaneous tradeoffs among three or more MOEs; it offers him the less complex, though still difficult task of assessing the tradeoffs between only two MOEs. It thus allows him to focus his attention on that part of his internal model of the important overall effectiveness relationships that relates to the two MOEs.

To facilitate the construction of the sequence of hypothetical outcomes, a tradeoff assessment tableau was devised. After the two real alternative programs have been selected and their MOEs evaluated, we inspect the successive pairs of values for each MOE. This allows us to determine the number of MOEs, m, such that the a_i's dominate the corresponding b_i's; the number of MOEs, q, such that the b_i's dominate the corresponding a_i's; and the number of MOEs, p, such that $a_i = b_i$.

We select the minimum of m and q (assume it is m) and rearrange the MOEs so that the first m consist of the case where $a_i > b_i$, the next q consist of the case where $a_i < b_i$, and the remaining p consist of the case where $a_i = b_i$. For specificity assume that $m = 2$, $q = 3$, and $p = 1$. We can now construct the tableau shown in Figure III-2. Note that if q is less than m, we can always switch names of the real outcomes (A to B and B to A) so that the same tableau form results.

Figure III-2 shows several interesting properties of this procedure. First, the maximum number of hypothetical outcomes, and thus the maximum number of tradeoff assessments, is $m+q-1$ (4 in this case). The minimum number of tradeoff assessments required is max(1, m-1) (1 in this case). The minimum number of tradeoff assessments would occur if c_2 were less than or equal to

b_2 (or in general if c_m were less than or equal to b_m). In such a case, B would dominate the (m-1)-th hypothetical outcome in at least q MOEs and not be dominated by any remaining MOE. Thus, B would be preferred to A.

MOEs / Alternative Programs	$a_i > b_i$		$b_i > a_i$			$a_i = b_i$
	x_1	x_2	x_3	x_4	x_5	x_6
(Real) A	a_1	a_2	a_3	a_4	a_5	a_6
(Hypothetical) A′	b_1	$\boxed{c_2}$	a_3	a_4	a_5	a_6
(Hypothetical) A″	b_1	b_2	$\boxed{c_3}$	a_4	a_5	a_6
(Hypothetical) A‴	b_1	b_2	b_3	$\boxed{c_4}$	a_5	a_6
(Hypothetical) A⁗	b_1	b_2	b_3	b_4	$\boxed{c_5}$	a_6
(Real) B	b_1	b_2	b_3	b_4	b_5	$b_6 = a_6$

FIGURE III-2 TRADEOFF ASSESSMENT TABLEAU

If, in fact, B is preferred to A we discover that fact anywhere from the (m-1)-th tradeoff assessment to the (m+q-1)-th tradeoff assessment. If, on the other hand, A is preferred to B or both are equally preferred we would discover that fact only after the (m+q-1)-th tradeoff assessment.

In Figure III-2 we can also note the relationship between any hypothetical outcome and each of the pair of real outcomes. We see that the first k MOEs of the k-th hypothetical outcome are equal in value to the MOEs of outcome B, and the last n-k-1 are equal to the MOEs of outcome A. Thus, the hypothetical outcomes can be readily constructed from A and B. The remaining MOE value to complete each hypothetical outcome is supplied by the tradeoff assessment of the DM. He accomplishes this without any thought as to whether or not there exists a real ED program with that outcome, or what its real cost may be. This is true since even if the program were feasible, he need never seriously consider implementing it. If it costs more than or equal to the budget, he will select either A or B, since A is as good and B may be better and neither costs more. If it costs less than the budget, he should advise his staff to find an improved alternative whose cost equals the budget, and he will select either the new alternative or B.

Pairwise tradeoff assessment information provides a local approximation to the marginal rate of substitution between two MOEs. If the marginal rate of substitution is independent of the outcome MOE levels, then the tradeoff ratio becomes a global estimate of the marginal rate of substitution. In this case, we have a linear indifference model of the DM's preference function.

The linear indifference model can be employed in conjunction with the tradeoff assessment procedure to analytically determine a most preferred outcome using a linear optimization procedure. The degree to which this outcome is truly the "most preferred" depends on how well the constant marginal rate of substitution assumption is satisfied.

The linear indifference model results in an objective function of the form

$$g(\underline{x}) = \sum_{i=1}^{n} \gamma_{ni} x_i \tag{3}$$

where n is the number of MOEs and x_i is the value of the i-th MOE outcome. The values of the s are obtained from the tradeoff assessment procedures that generate the values of the c´s in Figure III-2. In particular, the tradeoff ratio between any adjacent pair of MOEs, as shown in the tableau in Figure III-2, is defined as

$$\gamma_{i+1,i} = \frac{b_i - a_{i+1}}{c_i - b_{i+1}} \tag{4}$$

The tradeoff ratio between the n-th and any arbitrary i-th MOE is defined as

$$\gamma_{ni} = \prod_{k=i}^{n-1} \gamma_{k+1,k} \tag{5}$$

for $i < n$, and is equal to 1 for $i = n$.

The outcome from among the set of alternative outcomes that maximizes $g(\underline{x})$ is the most preferred outcome according to the linear indifference model. The optimization procedure is graphically illustrated for a two dimensional case in Figure III-3. The value of γ_{21} corresponds to the negative of the slope

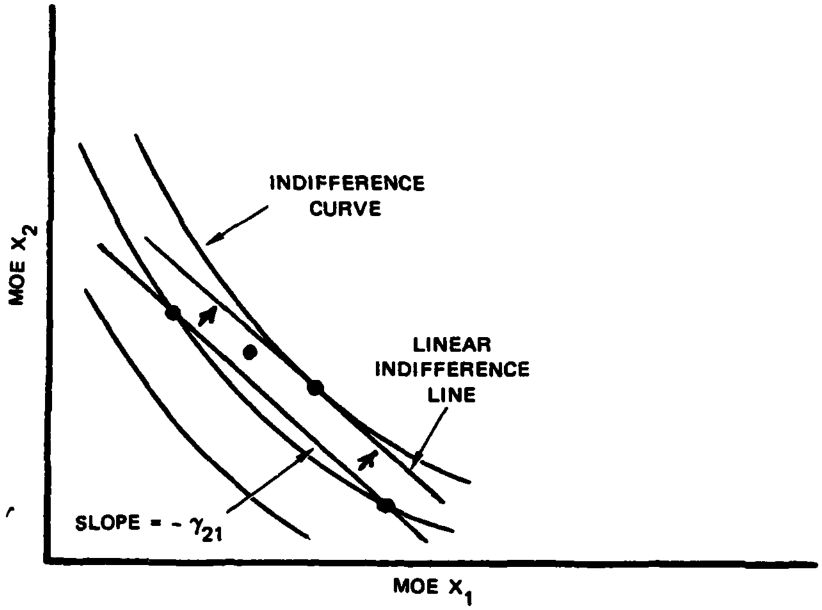


FIGURE III-3 GRAPHICAL OPTIMIZATION

of the linear indifference line. Optimization corresponds to sliding this line up and to the right until it reaches the last outcome. This is the "most preferred" outcome. In higher dimensions the γ s define a hyperplane, and optimization is achieved by sliding the hyperplane in the direction of increasing MOE values to the last outcome.

The steps and procedures of the Basic RA Method are the following:

(1) Select alternative pair. Initially select the two potentially most preferred alternative outcomes based on the subjective assessment of the DM. On subsequent iterations through Step 1, select the current highest-ranking alternative and one other potentially most preferred alternative from among the remaining alternatives.

(2) Reorder MOEs. Reorder the MOEs so that the minimum number of dominated MOEs between the two alternatives are at the beginning of the sequence of MOEs, and the remaining reverse-dominated MOEs are listed next, followed by the remaining equal-value MOEs (if any).

(3) Construct tableau. Construct the tradeoff assessment tableau so that the top alternative dominates the bottom alternative in the first MOE.

(4) Perform tradeoff assessment. Obtain the DM's response between the appropriate pair of MOEs within the tableau. Each response completes the construction of a hypothetical outcome.

(5) Test for dominance. After the minimum number of tradeoff assessments (m-1) have been completed, determine whether the bottom alternative completely dominates the last hypothetical alternative. If it does, the bottom alternative is ranked as more preferred than the top alternative. If unranked alternatives remain,

we proceed to Step 1 or Step 6 at the option of the DM. Otherwise, we have completed the procedure and the most preferred alternative has been identified. If, on the other hand, dominance has not yet occurred, further tradeoff assessments are required and we return to Step 4. After the maximum number of tradeoff assessments (m+q-1) have been completed, the two alternatives are ranked by comparing the last MOE involved in the tradeoff assessment. Again, we proceed to Step 1 or Step 6 if unranked alternatives remain.

(6) Complete the tradeoff assessments (optional). If a complete set of tradeoff ratios has not been established, the γ_{ni}'s in Eq. 3 are not all known and the linear function cannot be optimized to determine the next alternative. However, at the option of the DM, the remaining tradeoff assessments (obtained according to Step 4) can be accomplished. The γ_{ni}'s, which are tradeoff ratios between the n-th or last MOE and the i-th MOE can then be computed according to Eq. 5.

(7) Perform linear optimization for next alternative selection. Select the next alternative by optimizing Eq. 3 over all remaining alternatives.

(8) Test for termination (optional). Ask the DM to carefully consider each of his most recent tradeoff ratios and the range of MOE values covered by the remaining alternatives. Determine whether he would modify any of these tradeoff ratios as a function of the MOE values within this range. If he would not, the most preferred outcome is the outcome that optimizes Eq. 3. At this point, the most preferred alternative has been

identified and we are done. If the DM indicates that his tradeoff ratios are not constant over the range of MOE values,proceed to Step 2 with the current highest-ranking alternative and the alternative obtained in Step 7.

IV NONLINEAR EFFECTS IN MOE TRADEOFFS

A. General

A portion of the Basic RA Method, described in the preceding chapter, is predicated on the assumption that the DM's indifference curves for each pair of MOEs is linear, at least in the region of decision. That is, the DM is willing to sacrifice K units of one MOE in order to gain one unit of the other MOE, where K is a constant within the region of possible alternative outcomes.

As described in Chapter III, the Basic RA Method consists of two parts; an iterative evaluation that eventually identifies a most preferred outcome from a set of alternative outcomes, and an analytical model that can be employed to accelerate the process of identifying a most preferred outcome. The analytical model portion of the RA method is based on the linearity assumption. This assumption is usually sufficient in cases where the region of decision is relatively small. However, as this region expands, the effects of nonlinearities in the DM's preference structure may become more and more significant.

This research then was directed toward examining the effects of such nonlinearities on the Basic RA Method and to identify procedures to be applied when the linearity assumption becomes unacceptable. The approach used was to first identify the types of indifference curves that may be encountered in practice and to establish a categorization scheme to be used to classify relationships between pairs of MOEs. The second step was to theoretically examine the implications that these nonlinearities will have on the use of the Basic RA Method. The final step was to develop a nonlinear indifference model and the procedures required to incorporate it into the RA method.

B. Categorization of Nonlinear Indifference Curves

The Basic RA Method is based on the assumption that the DM's preference pattern between a pair of MOEs results in a linear indifference curve. In some practical applications, this may well be the case. However, there are many circumstances where the DM's preference pattern may vary from this norm. In order to represent these variations in a convenient manner, a categorization scheme was developed to classify the types of nonlinearities that one may encounter in practical situations. The categories that were selected are as follows:

- Threshold
- Convexity
- Complementarity
- Disutility
- Proportionality

These categories are not necessarily exclusive, in the sense that an indifference curve could, over the complete decision region, exhibit several of the characteristics representing these categories. However, in a local portion of the decision region, it is assumed that, at most, only two of the categories hold. This commonality of characteristics will become apparent in the descriptions of the various categories that follow. In these descriptions, it is assumed that an increase in a value of a MOE is beneficial in the eyes of the DM. That is, more of a MOE is better than less. This convention precludes the use of some common MOEs, such as 'response time', where the lesser value is preferred to the greater. However, these MOEs can be included merely by assuming their reciprocal values.

Before discussing each of these nonlinear categories, we review the linearity assumption case. Linearity represents the case where there is a constant tradeoff between two MOEs. That is, the DM is willing to sacrifice K units of one MOE in order to gain one unit of the other MOE, where K is a constant. This case is illustrated in Fig. IV-1. This case, which is the basis for

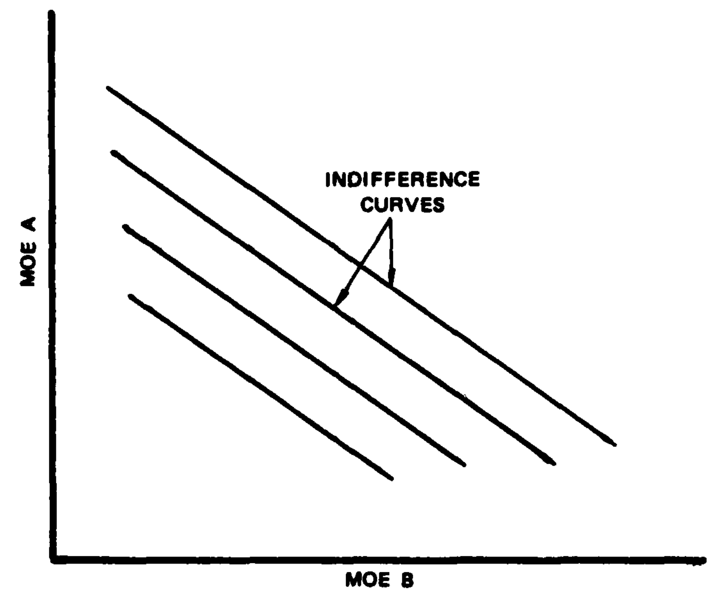


FIGURE IV-1 LINEARITY CASE

the Basic RA Method, provides a local approximation for the more prevalent convex case, although it may actually be appropriate in certain cases. For example, in the tradeoff between initial investment cost and annual operations and maintenance Navy (O&MN)

cost, the DM may be inclined to be indifferent to a fixed ratio between these costs. That is, for every million dollars of investment, the DM may be indifferent to an O&MN cost of $75 K per year, amortized over a period of ten years.

1. Threshold

The threshold categorization factor represents the case where a minimum amount of one or both of the MOEs is always required. This case is illustrated in Fig. IV-2, where MOE B

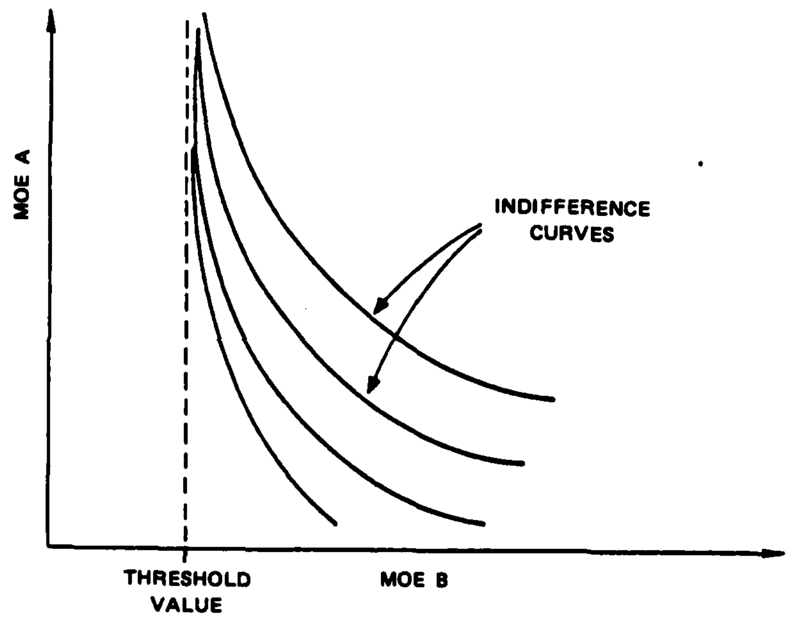


FIGURE IV-2 THRESHOLD CASE

exhibits this threshold characteristic. As an example, MOE B could represent the fuel resupply rate for a ship deployed at sea for an extended period of time. The threshold value would denote the minimum rate at which the ship could just manage to function

in its operational environment. Increases in this rate would allow the ship more flexibility in its ability to maneuver, which could reduce the requirements imposed on the ship's function associated with MOE A that represents, for example, the ship's firepower. The indifference curves illustrated in the figure are normally convex leading into the threshold barrier. This implies that the closer the restricted MOE gets to the threshold barrier, the greater the increase in the other MOE required to offset a small decrease in the restricted MOE.

2. Convexity

The convexity categorization factor represents the case where the DM is willing to accept a loss in one MOE that is offset by an increase in the other MOE, and the amount of offsetting increase must be larger as the amount of the first MOE gets smaller. This case is illustrated in Fig. IV-3. This case

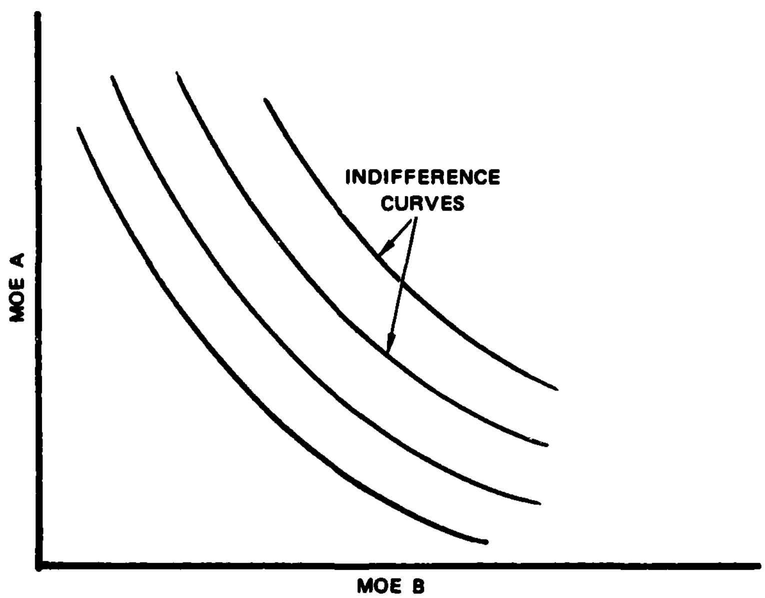


FIGURE IV-3 CONVEXITY CASE

represents the usual norm for indifference curves and is equivalent to the increasing marginal rate of substitution assumption. A good example of this is the tradeoff between a ship's maneuverability and its firepower, as mentioned previously. Increasing maneuverability will reduce the requirements for firepower, but at a decreasing rate. The ship, by increasing its maneuverability, may decrease its susceptability to the submarine threat, but it may still be as susceptible to the air threat as it was before. Hence, its firepower requirements against the submarine threat may be reduced somewhat, but it still must maintain the same firepower requirements against the air threat. Thus, its total firepower requirements may be reduced somewhat, but not proportionately to its initial requirements.

3. Complementarity

The complementarity categorization factor represents the case where the DM's indifference curve is either convex or linear, but there exists a point of radical change in this indifference. This case is illustrated in Fig. IV-4. What this case implies is that the DM has a specific balance point between the two MOEs under consideration, and any deviation from this point is one MOE requires a markedly different tradeoff in the other MOE than would be the case if the MOEs were reversed. As an example of such a case, consider the possible tradeoff between equipment-on-hand and spare parts inventory. For a specific level of operations, there may exist an optimal balance point which specifies a certain number of spare parts required for a desired number of units of equipment. A deficiency in the number of spare parts can be offset by the addition of a specific number of units of equipment. However, a lack of the same number of units of equipment can only be compensated for by a much larger increase in spare parts than was deficient in the former case.

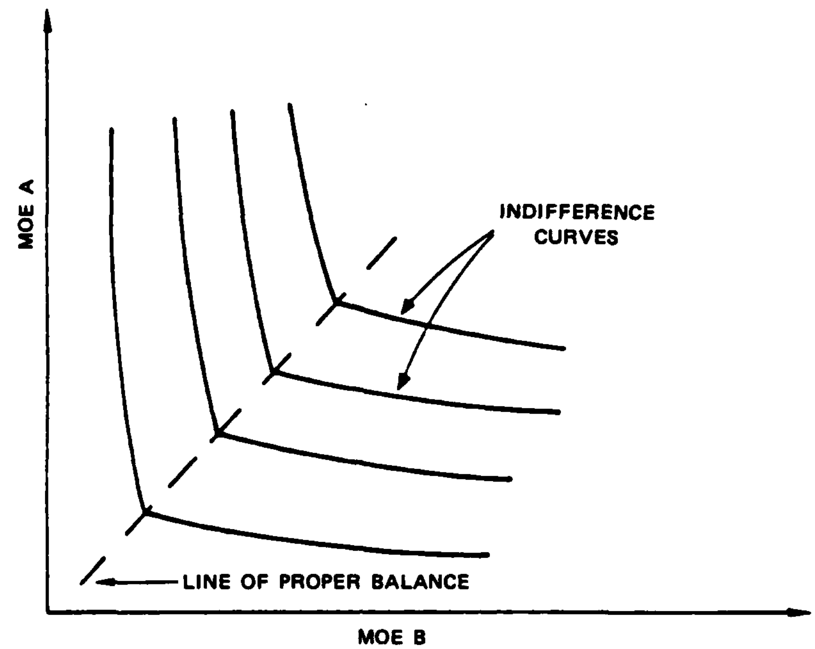


Figure IV-4 COMPLEMENTARITY CASE

'. Disutility

The disutility categorization factor is similar in nature to the complementarity categorization factor, with the exception that, on either side of the balance point, an abundance in one MOE can only be offset by a proportional abundance in the other MOE. This case is illustrated in Fig. IV-5. What this case implies is that the DM has a preference for a proper balance between the two MOEs and an increase in one MOE requires some proportional increase in the other M to satisfy the DM's preference criteria. As an example, consider an intermediate storage point such as a warehouse facility. The proper balance

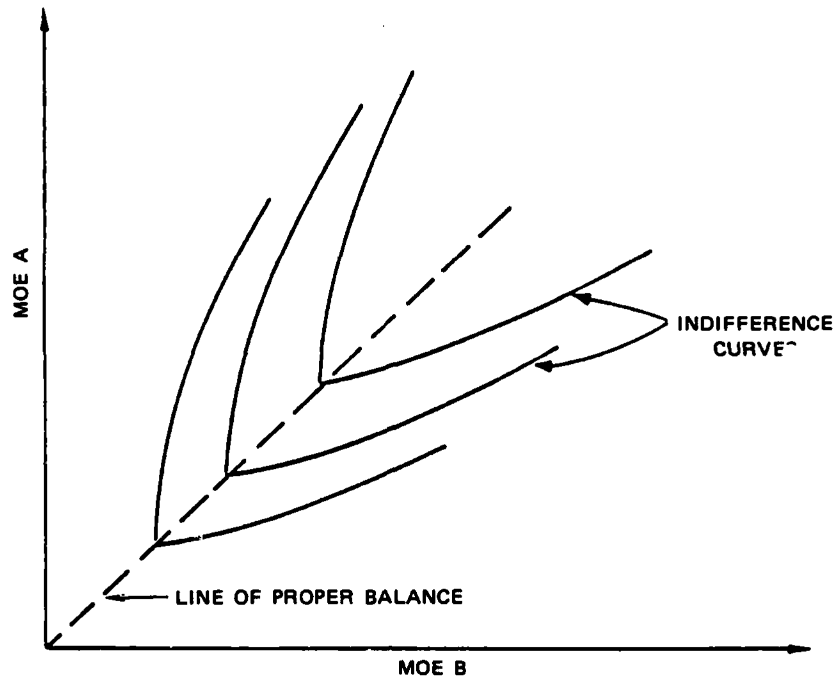


FIGURE IV-5 DISUTILITY CASE

point could be the warehouse being full to capacity with incoming supply metric tonnage exactly equaling outgoing supply metric tonnage. Any increase in the incoming supply rate could only be tolerated if the outgoing supply rate is also increased, and vice versa. If the former case were to hold with no increase in the outgoing supply rate, then the warehouse capacity would have to be increased or supplies would have to be returned to the sender. In the latter case where the outgoing supply rate exceeds the incoming supply rate, the warehouse would not be used to its full capacity and eventually would become nearly depleted, with the outgoing supply rate having to be reduced resulting in unused resources on the outgoing supply operation.

5. Proportionality

The proportionality categorization factor is actually an extreme case of the disutility categorization factor. This implies that the DM's indifference curve is actually a single point, the point of proper balance between the two MOEs. In the sample cited in the disutility case above, the DM cannot tolerate an imbalance in the incoming and outgoing supply rates. An increase in one supply rate can only be accompanied by an increase in the other supply rate and this balance point would then represent a higher level indifference point.

C. Implications of the Nonlinear Categories

The basis for the previous research that resulted in the Basic RA Method explicitly considered the cases of linearity and convexity. The implications of convexity on the Basic RA Method was that an iterative sequence of linear indifference models would converge to the identification of the most preferred outcome. The implications of the remaining nonlinear-categories:

- Threshold
- Proportionality
- Disutility
- Complementarity

are discussed below.

1. Threshold

The principal implication of the threshold case is that the threshold values must be identified, and this information incorporated into the alternative ED program selection process. Thus, any alternative outcome that has an MOE value below the required threshold should be discarded or the ED program modified to increase that MOE above the threshold. Referring to Figure IV-2 we see that this is equivalent to a translation of the outcome space along the MOE B axis. When this is done we see that the threshold case reduces to one of the remaining nonlinear cases.

Since the RA Method does not deal with the selection of alternatives, we have not explicitly considered these effects in the RA method development.

2. Proportionality

Proportionality implies that there is a strong functional relationship between pairs of MOEs. An indifference curve becomes a point, and the family of indifference curves becomes a line (or curve) of MOE proportions. Again, the principal implication of proportionality is on the selection of alternative ED programs. Obviously, the disutility for deviations from the proper proportion will be so great that outcomes whose MOEs do not satisfy this proportion should be discarded or modified.

A second important implication of proportionality is that the dimensionality of the MOE space can be reduced. For example, if MOEs A and B are proportionally related whereas A and C, and B and C are not, the tradeoff between A and C implies a unique tradeoff between B and C, and vice versa. If all alternative outcomes have the proper proportion between A and B, then the issue of finding a most preferred alternative will be decided by tradeoffs between other MOEs and either A or B.

3. Disutility

The disutility case is somewhat similar to the proportionality case in that changes of outcome MOEs away from the line of proper balance (see Figure IV-5) imply a rapid falloff in utility. Inspection of Figure IV-5 shows that if one started from an outcome on the line of proper balance an increase in both MOEs is required just to remain indifferent. But this implies a higher budget ED program. If feasible, a clearly better way to allocate

that higher budget is to increase the MOEs along the line of proper balance. This will result in a net increased utility. In general, then, the selection of alternative Ed programs should be as close to the proper balance as possible.

We conclude that the disutility case has effectively the same implications on the RA method as the proportionality case.

4. Complementarity

The complementarity case also has a certain degree of coupling between pairs of MOEs, but the effect is much reduced from the disutility case. In particular we note that improvements in outcome utility can still consist of a tradeoff between the MOEs. This tradeoff however is not constant and the tradeoff ratio for improvement depends on which MOE is increased or decreased. This latter property is also true of the convexity case.

The knee in the indifference curves of the complementarity case suggests that a piecewise linear model would more accurately model the indifference function. Referring to Figure IV-4, a piecewise linear model would consist of a linear tradeoff model with a given tradeoff ratio to the right of the line of proper balance, and a second linear tradeoff model with a different tradeoff ratio to the left of the line of proper balance. In higher dimensions, a set of hyperplanes pieced together in the proper way constitutes the piecewise linear-model.

This type of model can be constructed using the same basic tradeoff assessment procedures as the Basic RA Method. However, for every pair of MOEs we would then require two types of tradeoff assessments. In one case the DM would decrease one MOE

in response to an increase in the other, and in the second case the DM would increase the first MOE in response to a decrease in the second MOE. In addition tradeoff assessments will be required to obtain information concerning the proper balance of the MOEs.

Based on the above considerations, we have developed an Extended RA Method that incorporates a piecewise linear indifference model.

V THE EXTENDED RA METHOD

A. Development of the Extended RA Method

The Extended RA Method is a piecewise linear extension of the linear indifference model. It is primarily designed to handle the first order nonlinear effect introduced by complementarity between and among MOEs. However, its utility also applies to the more general cases of increasing marginal rate of substitution.

The principal concept of the piecewise linear model is to contruct a nonlinear objective function that consists of linear indifference segments. This concept is illustrated in Figure V-1.

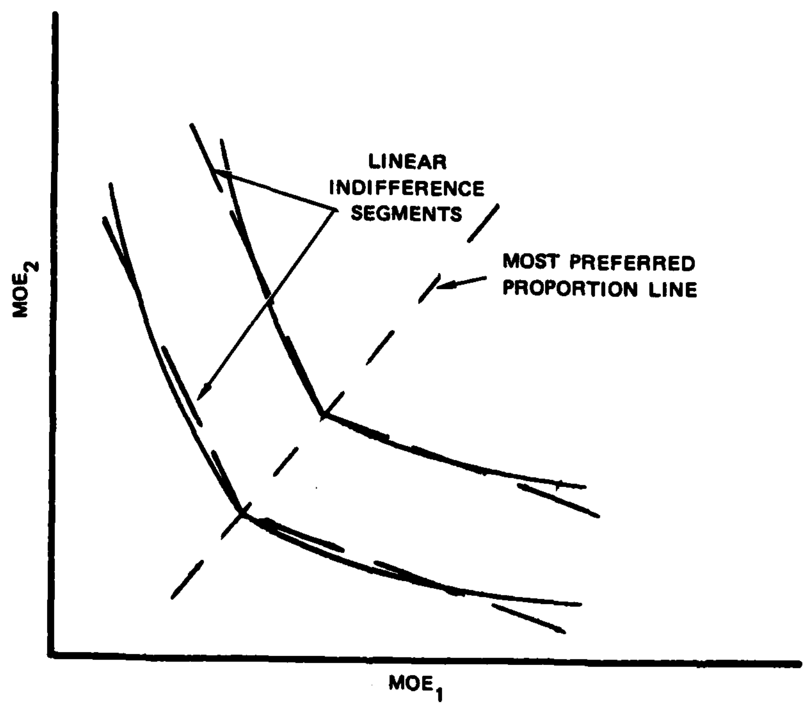


FIGURE V-1 PIECEWISE LINEAR INDIFFERENCE CURVES

The indifference structure illustrated in this figure is the complementary MOE case where, in general, a certain incremental proportion between the two MOEs is most preferred. However, incremental increases in either MOE without corresponding increases in the other still has some increase in utility, and therefore, preference. When these conditions prevail, the indifference functions have the shape illustrated in Figure V-1. The most preferred proportion between the two MOEs is shown by the dashed line. The indifference function shows that if one considers an outcome that has a most preferred proportion between MOE_1 and MOE_2, then a small decrease in either MOE requires a large increase in the other MOE. Conversely, given a large increase in either MOE, only a small decrease in the other can be "tolerated" to make the DM indifferent. Graphically, this situation is characterized by indifference functions that have a "knee" at the line of "most preferred" proportion.

A first order nonlinear approximation of each indifference curve is to piece together two linear functions at the line of most preferred proportion. The optimization problem to find the most preferred outcome then involves sliding the two associated slope lines along the line of most preferred proportion until one of the slope lines encounters the last outcome.

The piecewise linear model requires more information from the DM than the linear model, as would be expected. However, the same basic tradeoff assessment procedures as used in the Basic RA Method are still appropriate. Also, it is necessary to distinguish between two types of tradeoff assessments depending upon whether one of a pair of MOEs is increased or decreased in the tradeoff assessment. These two types of tradeoff assessments will be called negative or positive tradeoff assessments according to the following definitions:

Negative Tradeoff Assessment. A negative tradeoff assessment occurs when a DM decreases one MOE in response to a given increase in the other MOE.

Positive Tradeoff Assessment. A positive tradeoff assessment occurs when a DM increases one MOE in response to a given decrease in the other MOE.

In addition to the tradeoff ratios produced by the negative and positive tradeoff assessments, we need to determine the vector that defines the most preferred proportion line. One way to specify this vector is to specify one point on the proportion line and a direction vector for the line. Each point can then be obtained as the sum of the specified point (or vector) and some multiple of the direction vector. The information required to determine the direction vector can also be obtained by the basic tradeoff assessment process as will be described shortly.

The piecewise linear objective function is derived in Appendix B in terms of the direction cosine representation of two vectors: the hyperplane vector and the most preferred proportion vector. The direction cosine representation for these vectors, $\underline{\alpha}$ and $\underline{\epsilon}$, are obtained by simply dividing each term of each vector by its vector magnitude. When this is done, the above vectors are interpreted as follows. The vector $\underline{\alpha}$ is the unit vector normal to a given hyperplane and the vector $\underline{\epsilon}$ is the unit vector defining the direction of the line of most preferred proportion. The dot product of any given outcome vector with $\underline{\alpha}$ gives the "effective" minimum distance of the outcome point from a hyperplane passing through the origin. The dot products of $\underline{\alpha}$ with $\underline{\epsilon}$ gives the projection of $\underline{\epsilon}$ along the $\underline{\alpha}$ direction respectively. This last function, evaluated for each hyperplane, provides the weighting functions in the objective function.

Consider the following objective function for the two-dimensional case, where the hyperplanes are lines with unit normal vectors $\underline{\alpha}^1$ and $\underline{\alpha}^2$ respectively:

$$g(\underline{x}) = \operatorname{Min}\left[\frac{g_1(\underline{x} - \underline{a})}{g_1(\underline{\varepsilon})}, \frac{g_2(\underline{x} - \underline{a})}{g_2(\underline{\varepsilon})}\right] \tag{6}$$

where

$$g_1(\underline{x}) = \underline{\alpha}^1 \cdot \underline{x} = \sum_{i=1}^{n} \alpha_i^1 x_i \tag{7}$$

and

$$g_2(\underline{x}) = \underline{\alpha}^2 \cdot \underline{x} = \sum_{i=1}^{n} \alpha_i^2 x_i \tag{8}$$

and the vector, $\underline{a}$, is an outcome on the most preferred proportion line. Generally, it will be the point at which the negative and positive tradeoff assessments are made and the corresponding tradeoff ratios computed. However, it can be any outcome along the most preferred proportion line since the terms in the brackets are only modified by an additive constant.

Graphically, $g(\underline{x})$ is interpreted as follows. The quantities $g_1(\underline{x})$ and $g_2(\underline{x})$ are the minimum distances from the origin of the slope lines passing through $\underline{x}$. The quantities $g_1(\underline{x}-\underline{a})$ and $g_2(\underline{x}-\underline{a})$ are the separations between parallel slope lines passing

through $\underline{x}$ and through $\underline{a}$. The terms $1/g_1(\underline{x})$ and $1/g_2(\underline{x})$ are weighting factors that measure the relative importance of a change in the slope line separations along one slope line compared to the other. These weighting factors tend to favor points on the most preferred proportion line.

For example, consider the slope lines, most preferred proportion line, and outcomes shown in Figure V-2. The value of

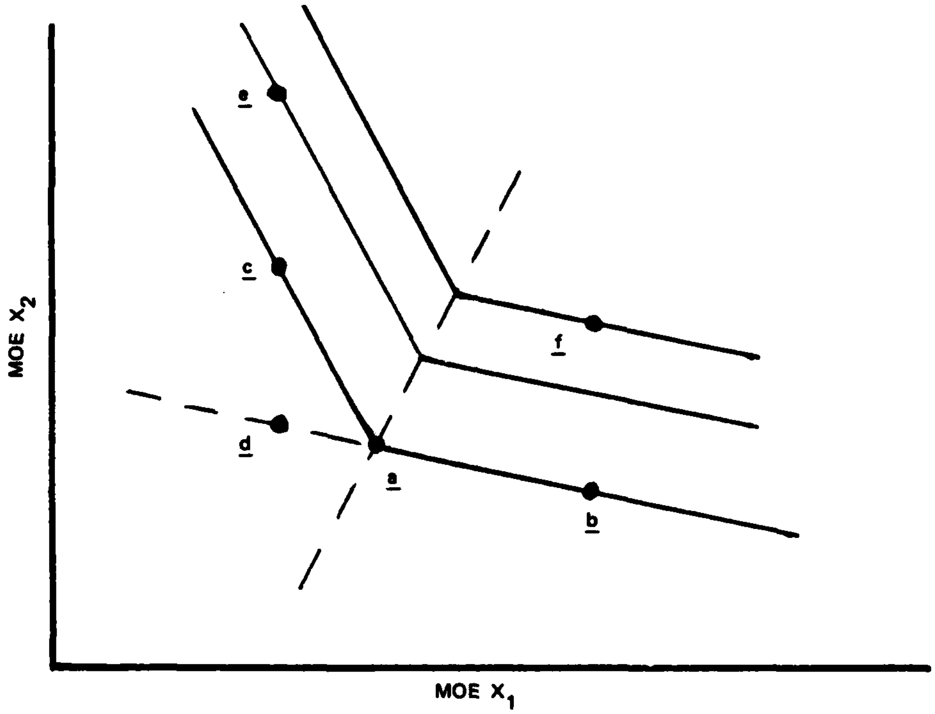


FIGURE V-2 PIECEWISE LINEAR GRAPHICAL OPTIMIZATION

$g_2(\underline{b}-\underline{a})$ is zero, and the value of $g_1(\underline{b}-\underline{a})$ must be greater than zero. Thus, $g(\underline{b})$ will be zero. Conversely, the value of $g_1(\underline{c}-\underline{a})$ is zero, and the value of $g_2(\underline{c}-\underline{a})$ is positive. Thus, $g(\underline{c})$ will also be zero. In fact, any point along the piecewise linear function in Figure V-2 will be zero, and therefore, equally preferred.

For point $\underline{d}$, $g_2(\underline{d}-\underline{a})$ is zero and $g_1(\underline{d}-\underline{a})$ is negative. Thus, $g(\underline{d})$ has a negative value relative to $\underline{a}$, $\underline{b}$, and $\underline{c}$. In a similar fashion, we can argue that both $\underline{e}$ and $\underline{f}$ have positive values relative $\underline{a}$, $\underline{b}$, and $\underline{c}$.

As illustrated in Figure V-2, $\underline{e}$ was constructed by adding an amount of MOE_2 to $\underline{c}$, and $\underline{f}$ was constructed by adding the same amount of MOE_2 to $\underline{b}$. Which ($\underline{e}$ or $\underline{f}$) has the greater value of g? The answer is provided by the value of the weighting functions. For this example, those weights will be such that $g(\underline{f}) > g(\underline{e})$. This means that a change in MOE_2 of a given amount starting from point $\underline{b}$ is more preferred than the same amount of change in MOE_2 starting from point $\underline{c}$. This result is consistent with the fact that $\underline{f}$ is closer to the most preferred proportion line than $\underline{e}$.

Generalizing the objective function to higher dimensional cases, the tradeoff slope lines become hyperplanes, and the objective function becomes

$$g(\underline{x}) = \min_{i=1,h} \left[\frac{g_i(\underline{x} - \underline{a})}{g_i(\underline{\varepsilon})} \right] \qquad (9)$$

where h is the number of hyperplanes established by the piecewise linear model. Each of these hyperplanes pass through the reference point $\underline{a}$.

The interpretation of the terms in the brackets is exactly the same as before, but couched in terms of hyperplanes. Thus, $g_i(\underline{x}-\underline{a})$ is the separation between $\underline{x}$ and $\underline{a}$ along a direction normal to the i-th hyperplane. The i-th hyperplane is defined by the direction cosine vector obtained by the i-th combination of tradeoff ratios.

Appendix A specifies how the set of h hyperplanes used for the piecewise linear model of the indifference function are determined from the tradeoff assessment data. The total number of MOE pairs that can be used for tradeoff assessment evaluations is equal to n(n-1)/2. However, since for each pair of MOEs we have both positive and negative tradeoff assessments, the maximum number of distinct tradeoff ratios is n(n-1).

Tradeoff ratio information can be encoded in vectors that have only two non-zero terms. For tradeoffs between the i-th and j-th MOEs, the two non-zero tradeoff ratio vector terms are the i-th and the j-th terms; all remaining terms are zero. Thus, these vectors have the form

$$\underline{v}(i,j) = (0, \ldots, \underset{\substack{\uparrow \\ \text{i-th}}}{-\gamma_{ij}}, \ldots, \underset{\substack{\uparrow \\ \text{j-th}}}{1}, \ldots, 0) \tag{10}$$

where γ_{ij} is the tradeoff ratio obtained from the negative tradeoff assessment of MOE_i relative to MOE_j. Also note that the tradeoff ratio always has a positive value. The tradeoff ratio obtained from a positive tradeoff assessment of MOE_i relative to MOE_j can be viewed as a tradeoff ratio obtained from a negative tradeoff assessment of MOE_j relative to MOE_i. Thus, we can define all the tradeoff ratio vectors in terms of equivalent negative tradeoff assessment information. The counterpart of $\underline{v}(i,j)$ then becomes $\underline{v}(j,i)$ and its j-th term is $-\gamma_{ji}$. These vectors represent sample points (within a scale factor) from the nonlinear indifference hypersurface we are attempting to model and from the piecewise linear hyperplanes we will use in our model.

The piecewise linear model is obtained by first finding all hyperplanes that each contain a combination of n-1 linearly independent tradeoff ratio vectors taken from the total set of n(n-1) vectors. We then select only those hyperplanes that are consistent with the increasing marginal rate of substitution assumption, and that include each tradeoff ratio vector in at least one hyperplane.

The hyperplanes obtained are also represented by vectors $\underline{w}$ which have a direction orthogonal to the hyperplane. In general, $\underline{w}$ need only be defined to within a scale factor. When normalized to a unit vector, $\underline{w}$ becomes a unique vector designated $\underline{\alpha}$.

The increasing marginal rate of substitution assumption is equivalent to a convexity assumption that translates into two conditions:

(1) For any given i and j, $\gamma_{ij} < 1/\gamma_{ji}$

(2) The dot product of each tradeoff ratio vector with any hyperplane vector used in the piecewise linear model, must be greater than or equal to zero. (It will equal zero if and only if the tradeoff ratio vector lies in the hyperplane under consideration).

The "inclusion condition" states:

Each tradeoff ratio vector must be contained in at least one hyperplane (i.e., $\underline{v} \cdot \underline{w} = 0$) that does not violate the convexity conditions.

The violation of the inclusion condition means that the tradeoff assessment data for that vector is inconsistent with the convexity condition. If this occurs, two options are available. The first is to reassess the tradeoff ratio until it is consistent with convexity. The second approach is to simply consider it as a "bad" datum, discard it, and form a limited piecewise linear indifference model with the remaining "good" data.

The maximum number of hyperplanes that can be formed from a set of s = n(n-1) vectors is given by

$$m = \frac{s!}{(n - 1)!(s - n + 1)!} \tag{11}$$

Most of these however, violate the convexity and inclusion conditions stated above and are thus not allowed in the model.

Appendix A shows that if the tradeoff ratios γ_{ki}, γ_{ij}, and γ_{kj} are all mutually consistent with the convexity assumption, then the selection of n-1 tradeoff ratio vectors to form a hyperplane cannot include pairs that are of the form $\underline{v}(k,i)$ and $\underline{v}(i,j)$ where $k \neq j$. This result requires that all vectors containing an arbitrary MOE index i that belong to an allowed set of n-1 linearly independent tradeoff ratio vectors, must have that index always appear in either the first position or the second position of the index doublet characterizing those vectors. Thus, the pair $\underline{v}(4,1)$ and $\underline{v}(1,2)$ is not allowed.

Unfortunately, we have not been able to derive a general analytical expression for the maximum number of allowed hyperplanes as a function of the dimensionality. However, if we let q denote this maximum, and examine several cases, we can construct Table V-1.

Table V-1
Maximum Number of
Allowed Hyperplanes

n	m	q
2	2	2
3	15	6
4	220	32

Appendix A also shows that if the tradeoff ratios γ_{ki}, γ_{ji}, γ_{jk}, and γ_{kj} are all mutually consistent with the convexity assumption, then the selection of n-1 tradeoff ratio vectors to form a hyperplane cannot include pairs of the form $\underline{v}(k,i)$ and $\underline{v}(j,i)$ where $k \neq j$, unless $\gamma_{kj} < \gamma_{ki}/\gamma_{ji} < 1/\gamma_{jk}$.

This numerical condition will work to further reduce the number of hyperplanes in the piecewise linear model to some number h that is less than q. We expect the degree of reduction of hyperplanes to be significant but cannot quantify it without specific numerical values.

To summarize, the negative and positive tradeoff assessments yield up to n(n-1) distinct tradeoff ratio values which can be represented as n(n-1) vectors. In general, there will be fewer tradeoff ratio vectors since certain MOE pairs may be linearly traded off. In the extreme, if all tradeoff assessments are linear, there will be only one half as many distinct tradeoff ratio values; however, only one linearly independent set of n-1 vectors will be found. Thus, there will be only one hyperplane for the piecewise linear model.

The number of hyperplanes allowed in the piecewise linear model is constrained first by the number of linearly independent sets of n-1 vectors that can be found. Secondly, the convexity and inclusion conditions disallow certain hyperplanes. Finally, numerical values of certain tradeoff ratios may disallow additional hyperplanes.

Given an allowed set of h hyperplanes, the reference outcome vector $\underline{a}$ at which tradeoff assessments are made, and the most preferred marginal proportion vector $\underline{\epsilon}$, the piecewise linear objective function, $g(\underline{x})$, is given by Eq. 9. Selection of a most preferred alternative ED program, consists of finding an outcome vector that maximizes $g(\underline{x})$. Since each outcome vector corresponds to an alternative ED program, we choose that corresponding ED program.

B. Procedures of the Extended RA Method

The previously developed Basic RA Method is based on a sequence of pairwise MOE tradeoff assessments, the relative ranking of pairs of outcome vectors (i.e., alternative ED programs), and an optional linear model optimization procedure.

The research reported herein extended the linear model to a piecewise linear form of a nonlinear model. The utilization of the piecewise linear model increases the burden on the DM in terms of the number of tradeoff assessments required for local indifference modelling, and for establishing the most preferred proportion line. However, the piecewise linear model does also include the linear model in the case that only one hyperplane is required. In the latter case, the most preferred marginal proportion line corresponds to the direction of the unit vector normal to the hyperplane. Thus, the tradeoff assessment load will reduce to the linear model load.

The extension of the indifference model to a piecewise linear one does not affect the ranking portion of the Basic RA Method. The ranking procedure is designed to lead to the selection of the most preferred alternative from among a given set whether the indifference function is approximately linear or highly nonlinear. Unfortunately it provides no model of the indifference structure that can be employed to reduce the tradeoff assessment load when considering a new set of alternatives, or when attempting to synthesize new alternatives that are likely to be highly preferred. Thus, a linear model evaluation procedure was also included in the Basic RA Method. Steps 6 and 7 in the procedures of the Basic RA Method provide for this evaluation. The Extended RA Method procedures are obtained by revising these two steps. The list of steps then becomes:

(1) Select alternative pair
(2) Reorder MOEs
(3) Construct tableau
(4) Perform tradeoff assessments
(5) Test for dominance
(6) Complete the tradeoff assessments (optional)
(7) Perform piecewise linear optimization for next alternative selection (optional)
(8) Test for termination (optional)

1. Tradeoff Assessment Requirements for Step 6

Step 6 consists of 3 tasks. First, a matrix of pairwise tradeoff ratios must be determined to characterize the indifference relationships. Secondly, these tradeoff ratios must be checked for consistency with the convexity conditions. Finally, additional tradeoff assessments are required to determine the direction of most preferred proportion line.

To construct a piecewise linear indifference model, the first task is to obtain sufficient tradeoff assessment information from the DM. The number of tradeoff assessments required will depend on the number of linear pairwise indifference relationships. The greater this number the fewer tradeoff assessments required. A linear pairwise indifference relationship implies that the negative and positive tradeoff assessments yield the same tradeoff ratio.

The maximum number of tradeoff assessments required is n(n-1) and the resulting tradeoff ratios can be arranged in a square matrix. The i-th row and j-th column of this matrix represents the tradeoff ratio of MOE_i relative to MOE_j for a negative tradeoff assessment. The diagonal elements of this matrix are all trivially equal to unity. For i greater than j, the entry is an equivalent negative tradeoff assessment result obtained from a positive tradeoff assessment. Its value is the reciprocal of the positive tradeoff assessment ratio. For example, if a negative and positive tradeoff assessment of MOE_i with respect to MOE_j yields tradeoff ratios of 0.6 and 0.9, respectively, $\gamma_{ij} = 0.6$ and $\gamma_{ji} = 1/0.9 = 1.11$. These values can be entered in the (i,j) and (j,i) positions of the matrix. If MOE_i and MOE_j were linearly substitutable, both negative and positive tradeoff assessments would yield the same value, and if $\gamma_{ij} = 0.6$, then we would infer that $\gamma_{ji} = 1/0.6 = 1.67$. In the nonlinear case, the increasing marginal rate of substitution assumption requires γ_{ji} to be less than 1.67 when γ_{ij} is equal to 0.6.

A matrix of tradeoff ratios for a 4-dimensional problem is shown in Table V-2.

Table V-2

TRADEOFF RATIO MATRIX

		MOE			
		1	2	3	4
MOE	1	1	γ_{12}	γ_{13}	γ_{14}
	2	γ_{21}	1	γ_{23}	γ_{24}
	3	γ_{31}	γ_{32}	1	γ_{34}
	4	γ_{41}	γ_{42}	γ_{43}	1

It turns out that if the tradeoffs are linear for an entire row or column in this matrix, then the remaining tradeoff ratios can be inferred. If only a portion of these are linear then some but not all of the remaining elements can be inferred. The inferential structure of the tradeoff assessments is defined in Table V-3 for 3,4, and 5-dimensional problems. The structure for higher dimensional problems can, in turn, be inferred from Table V-3.

Table V-3

TABLE OF TRADEOFF INFERENCES

n	(1,2)	(1,3)	(1,4)	(1,5)	(2,3)	(2,4)	(2,5)	(3,4)	(3,5)	(4,5)
5	L	L	L	L	L	L	L	L	L	L
	L	L	L	NL	L	L	NL	L	NL	NL
	L	L	NL	NL	L	NL	NL	NL	NL	?
	L	NL	NL	NL	NL	NL	NL	?	?	?
	NL	NL	NL	NL	?	?	?	?	?	?
4	L	L	L		L	L		L		
	L	L	NL		L	NL		NL		
	L	NL	NL		NL	NL		?		
	NL	NL	NL		?	?		?		
3	L	L			L					
	L	NL			NL					
	NL	NL			?					

L - Linear

NL - Nonlinear

To use Table V-3 we select a row or column of the matrix of tradeoff ratios for evaluation and rearrange rows and columns by MOE reordering to move the desired elements to the first row. We then elicit both negative and positive tradeoff assessment information from the DM and insert values in the first row and column of the matrix. Linear tradeoffs are revealed whenever $\gamma_{ij} = 1/\gamma_{ji}$. We can then reorder MOEs such that all linear tradeoffs are moved to the left in a row, or to the top in a column. At this point we can refer to Table V-3 and find the appropriate case in one of the leftmost blocks.

Assume that the problem is 3-dimensional. The lower leftmost block shows that if the (1,2) and (1,3) tradeoffs are linear, we can infer that the (2,3) tradeoff will be linear. If both are nonlinear, we can infer nothing about the (2,3) tradeoff. If one is linear and the other is not, we can infer that the (2,3) tradeoff must also be nonlinear. The various inferences we can make for the 4 and 5-dimensional cases are also shown in Table V-3.

The results in Table V-3 can readily be extended to 6-dimensions by adding a new first row consisting of all L's to the leftmost 5-dimensional block, and a fifth column with L for the first element and NL's for the remaining elements. We can then fill in the blocks to the right by using the 3-dimensional results for the various 3-dimensional subspaces of the 6-dimensional case. For example, the fact that the (1,5) and (1,6) tradeoffs are linear infers that the (5,6) tradeoff will be linear. If (1,5) is linear and (1,6) is nonlinear, we infer that (5,6) tradeoff will be nonlinear.

In addition to these inferences characterizing the type of tradeoffs, we can also infer the value of the tradeoff ratios for those MOE pairs that we infer are linear. For example, if we know that the (1,2) and (1,3) tradeoffs are linear and have values γ_{12} and γ_{13}, we infer that the (2,3) tradeoff will be linear and will have a value of $\gamma_{23} = \gamma_{13}/\gamma_{12}$. Furthermore, $\gamma_{32} = \gamma_{12}/\gamma_{13}$. Thus the linear inferences reduce the tradeoff assessment load.

After evaluating the first row and column of tradeoff ratios, and filling in all the inferential information, the additional tradeoff assessment requirements are defined by the sub-matrix obtained by deleting all rows and columns from the top and left, respectively, whose elements have all been evaluated. The above procedure is repeated recursively until all elements have been evaluated.

Given a tradeoff ratio matrix we can next test for consistency of the values with the convexity conditions. The requirement for consistency is that the value of any given matrix element be bounded as required by Eq. A-22 in Appendix A. If any bound is violated, the tradeoff ratios that should be modified, and the minimum amount they should be modified by, can be inferred from Eq. A-22. If a consistency violation cannot be resolved, the remaining procedures can still be applied after excluding the inconsistent tradeoff ratio vector. This results in a piecewise linear model based on reduced tradeoff assessment information. The appropriateness of employing such a limited piecewise linear model has not been studied in this research effort. It will undoubtedly be dependent on specific cases and will require a judgemental decision.

The final task in step 6 is to perform the tradeoff assessments required to determine the most preferred proportion vector. This process is described in Appendix C. If we have a strictly linear case where all MOE tradeoffs are constant, this vector is not defined, and an arbitrary vector can be assumed. This is true since, in this linear case, there will be only one hyperplane, and the role of the weighting functions in the objective function will no longer be applicable. A convenient value for $\underline{\epsilon}$ in this case is to set it equal to the hyperplane vector $\underline{\alpha}$.

The evaluation of $\underline{\epsilon}$ requires n-1 additional tradeoff assessments between nonlinearly related MOE pairs. If a nonlinear tradeoff exists between any pair of MOEs, we are guaranteed by inference that there will be at least n-1 nonlinearly related MOE pairs from which we can perform n-1 independent tradeoff assessments. This is readily shown by the following argument. Assume that MOEs k and j are nonlinearly related. Then for any i not equal to j or k either the (i, k) or (i, j) MOE pairs are nonlinearly related according to our table of inferences (see Table V-3). Since there are n-2 values of i, there will be n-1 nonlinearly related pairs.

Any set of linearly independent MOE pairs can be chosen for these tradeoff assessments. As discussed in Appendix C, these tradeoff assessments involve the construction of an outcome of higher preference than $\underline{a}$ by increasing a single MOE. A positive tradeoff assessment is then performed by decreasing that MOE back to the level of $\underline{a}$, and asking the DM how much the second MOE must be increased to compensate for the reduction. For an (i,j) tradeoff, the computed tradeoff ratio yields γ'_{ij}. These tradeoff ratios plus the matrix of indifference tradeoff ratios illustrated in Fig. V-2 are required as input to the equations of Appendix C for evaluating $\underline{\epsilon}$.

2. Piecewise Linear Optimization Model for Step 7

After the matrix of tradeoff ratios and the vector representing the direction of most preferred proportion have been evaluated, the next task is to determine the vectors representing the allowed hyperplanes. The procedure for determining the hyperplanes consists of 7 steps.

1. Form the tradeoff ratio vectors defined by Eq. 10.
2. Select a set of n-1 linearly independent tradeoff ratio vectors.
3. Solve the set of n-1 equations formed by setting the dot product of each tradeoff ratio vector with the hyperplane vector.
4. Test all remaining tradeoff ratio vectors by computing their dot product with the hyperplane vector.
5. Discard the hyperplane if any of the dot products in step 4 are less than zero. Return to step 2 until all sets of n-1 vectors have been processed.

6. Retest all tradeoff ratio vectors by computing their dot products with all allowed hyperplane vectors. If there exists at least one zero dot product for each tradeoff ratio vector, the set of allowed hyperplanes is complete; otherwise perform the next step.
7. Those tradeoff ratio vectors failing the test in step 6 are inconsistent with the convexity assumption. The tradeoff ratio in these vectors should be reevaluated (with the assistance of the inferred tradeoff ratio bounds specified in Appendix A) and the process repeated from step 2 for those hyperplanes affected.

VI ILLUSTRATIVE EXAMPLE RA PROBLEM

A. Problem Description

The example selected to illustrate the use of the Extended RA Method addresses a standard inventory and maintenance problem at a large data processing center. It is assumed that the DM has at his disposal the analytical tools that can provide him with the optimal balance of units of equipment (central processors, power supplies, storage devices, printers, etc.) to insure specific levels of operations. For a specific required level of operation, say 90% assurance of continuous operation, the optimal balance consists of 30 units of equipment (assuming some redundancy), 75 spare modules, and 4 maintenance personnel. For any specified level of operation, the DM has strong feelings about maintaining the desired balance. However, he is willing to sacrifice, to some extent, a loss in one of the items at the expense of a gain in one or both of the other items.

The DM's indifference structure between equipment-on-hand and spare modules is complementary in nature, as is also his indifference structure between equipment-on-hand and maintenance personnel. On the other hand, he is linearly indifferent between spare modules and maintenance personnel. The applicable pairwise indifference curves representing three different levels of operations (80%, 90%, 95%), for this example, are shown in Figures VI-1, VI-2, and VI-3.

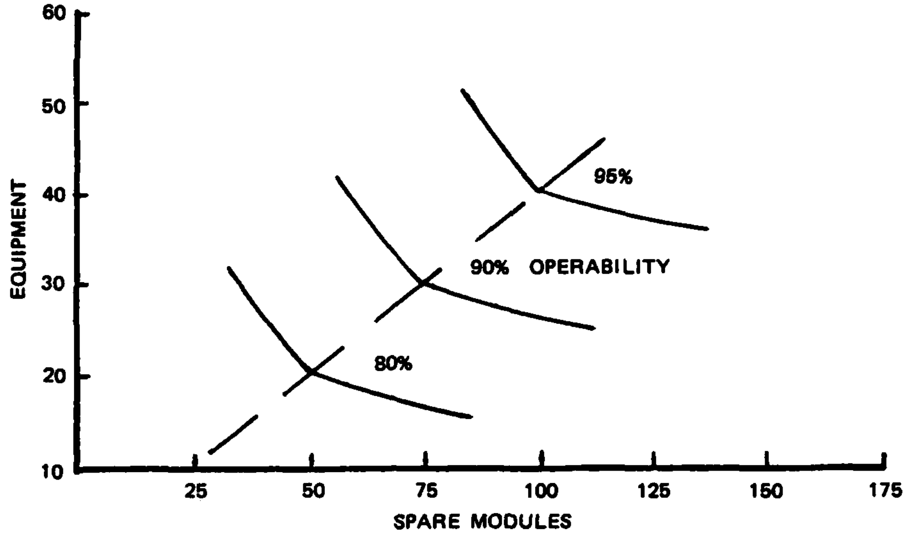


FIGURE VI-1 INDIFFERENCE CURVES (EQUIPMENT vs. SPARE MODULES)

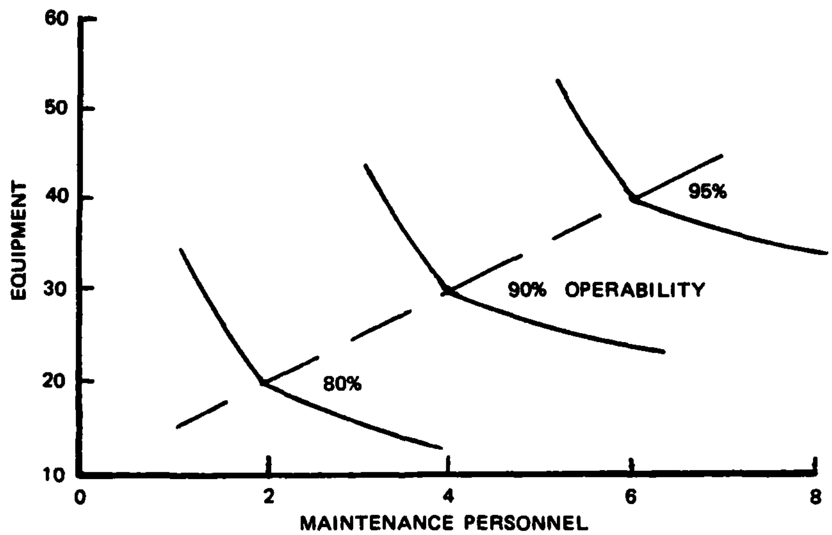


FIGURE VI-2 INDIFFERENCE CURVES (EQUIPMENT vs. MAINTENANCE PERSONNEL)

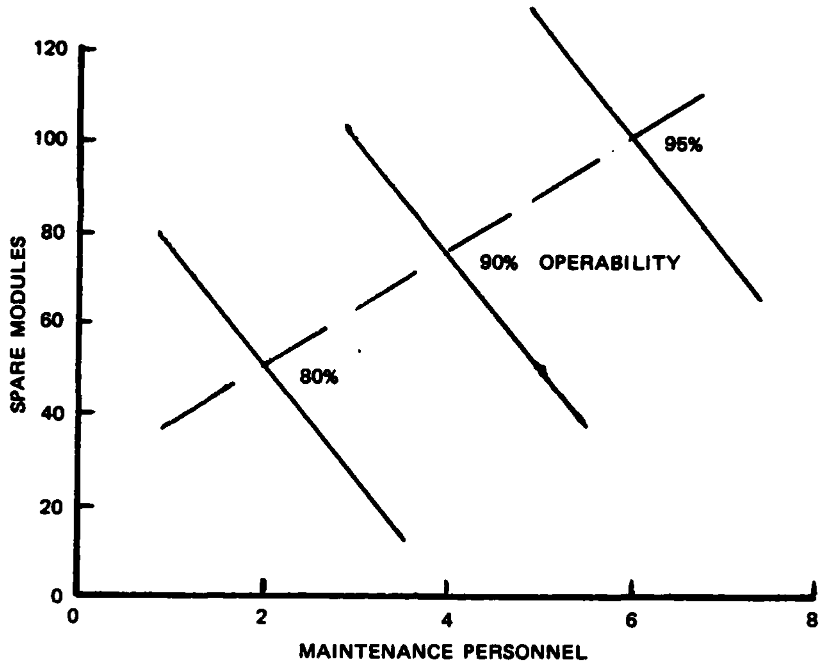


FIGURE VI-3 INIDFFERENCE CURVES (SPARE MODULES vs. MAINTENANCE PERSONNEL)

We assume that the indifference curves shown in these figures are available to us only for the purpose of this illustrative problem, and serve as a surrogate for the DM. Thus, any required tradeoff assessment can be accomplished by referring to these curves and reading off the equivalent responses of the DM. In addition, we will only illustrate portions of Steps 6 and 7 in the Extended RA Method procedures to show how the piecewise linear model is formed from the tradeoff information.

B. Tradeoff Assessment Information

The piecewise linear indifference model requires the identification of an outcome vector which may or may not correspond to a given alternative ED program, but that lies on the line of most preferred marginal proportion. It is in the neighborhood of this point that the DM has differing tradeoff ratios for the negative and positive tradeoff assessments. In addition to the evaluation of these tradeoff assessments, we also need to evaluate the most preferred proportions desired by the DM.

The first task is then to identify an outcome vector, $\underline{a}$, on the line of most preferred proportion. Assume that we focus on obtaining a 90% system operability figure. The DM responds that he feels that the most preferred proportion at that level of system operability is to have 4 maintenance personnel, 75 spare modules, and 30 units of equipment. We will designate the three MOEs as follows:

x_1 = Maintenance personnel

x_2 = Spare modules/15

x_3 = Equipment/5

Note than an arbitrary scaling of MOEs x_2 and x_3 has been performed to bring the numerical values within closer agreement. The vector $\underline{a}$ then becomes

$$\underline{a} = (4,5,6) \tag{12}$$

Starting at the reference point $\underline{a}$, we initially determine the negative tradeoff assessment ratios for x_1 with respect to x_2 and then with respect to x_3. Referring to Figures VI-1 through VI-3, we infer that the following tradeoff ratios are obtained (after MOE scaling):

$$\gamma_{12} = 0.6$$

$$\gamma_{13} = 0.35 \tag{13}$$

To obtain these values we assume that we asked the DM to decrease x_1 in response to an increase of x_2 of 1 scaled unit, and then in response to an increase of x_3 of 1 scaled unit.

Performing the positive tradeoff assessments we obtain tradeoff ratios of 0.6 and 1.43. Since the negative and positive tradeoff assessment ratios of x_1 with respect to x_2 are equal, we have a linear pairwise indifference relationship betweeen x_1 and x_2. The equivalent negative tradeoff ratios for the positive tradeoff assessment ratios are:

$$\begin{aligned} \gamma_{21} &= 1/0.6 = 1.67 \\ \gamma_{31} &= 1/1.43 = 0.7 \end{aligned} \tag{14}$$

The tradeoff ratio matrix at this point is:

1	0.6	0.3
1.67	1	NL
0.7	NL	1

The tradeoff between x_2 and x_3 must be nonlinear by inference using Table V-3. Furthermore, the convexity and inclusion conditions require that the value of γ_{23} must be equal to $\gamma_{21}\gamma_{13}$, according to Eq. A-24 in Appendix A. This is true since the

pairwise tradeoff between MOEs 1 and 2 is linear. Similarly the value of γ_{32} must be equal to γ_{31}/γ_{21} for the same reason. These values are 0.583 and 0.42, respectively. Note that the pairwise tradeoff between MOEs 2 and 3 are nonlinear since γ_{23} is not equal to $1/\gamma_{32}$. The tradeoff ratio matrix can now be updated to:

1	0.6	0.35
1.67	1	0.583
0.7	0.42	1

As a consistency check, we may ask for a tradeoff assessment between MOEs 2 and 3. If the DM is consistent with the convexity assumption we will obtain tradeoff ratios equal to these values as can be inferred from Figures VI-2 through VI-3. For the moment, assume we do obtain consistent results.

C. Most Preferred Proportion Determination

To determine the vector $\underline{\epsilon}$ that represents the direction of most preferred proportion, we must perform two (i.e., n-1 = 2) more tradeoff assessments as described in Appendix C. These tradeoff assessments will be between the MOE pairs x_2 and x_3, and between x_1 and x_3. In general we could have selected other MOE pairs involving tradeoffs between x_1 and x_2. However, in this example, the indifference function between x_1 and x_2 is linear, and Eq. C-11 will be indeterminate.

The required tradeoff assessment between MOE x_2 and x_3 proceeds by constructing a new outcome vector $\underline{b}$ from $\underline{a}$ by setting $b_3 = a_3 + 1$. The value 1 is in scaled units. Outcome $\underline{b}$ should be preferred to $\underline{a}$ by our assumption of "good" MOEs. We then ask the

DM how much we would have to increase MOE x_2 from a_2 (note that $a_2 = b_2$) in order to compensate for a decrease of x_3 from $a_3 + 1$ to a_3. The response determines a new outcome vector $\underline{c}$ that differs from $\underline{a}$ only in the value of x_2, and that is equally preferred to $\underline{b}$. The tradeoff ratio $_{32}$ is then computed from

$$\gamma'_{32} = \frac{b_3 - a_3}{c_2 - a_2} \tag{15}$$

Referring to Figure VI-1 we infer that the DM's response will be to set c_2 equal to a_2 + 1.32 scaled units of x_2. The value of γ'_{32} then becomes equal to 0.76. A similar tradeoff assessment between x_1 and x_3 yields a value of γ'_{31} equal to 1.59. Eq. C-11 yields

$$\begin{aligned} \tan \phi_{31} &= 1.0 \\ \tan \phi_{32} &= 0.84 \end{aligned} \tag{16}$$

Using Eq. C-12 yields the value of $\underline{\epsilon}$.

$$\underline{\varepsilon} = (0.608,\ 0.511,\ 0.608) \tag{17}$$

D. Hyperplane Determinations

The next task is to form the six tradeoff ratio vectors. These will be:

$$
\begin{aligned}
\underline{v}(1,2) &= (-0.6, & 1, & \quad 0) \\
\underline{v}(1,3) &= (-.35, & 0, & \quad 1) \\
\underline{v}(2,1) &= (\ 1, & -1.67, & \quad 0) \\
\underline{v}(2,3) &= (\ 0, & -0.58, & \quad 1) \\
\underline{v}(3,1) &= (\ 1, & 0, & \quad -0.7) \\
\underline{v}(3,2) &= (\ 0, & 1, & \quad -0.42)
\end{aligned} \tag{18}
$$

The selection of sets of vector pairs (i.e., n-1 is equal to 2 for this 3-dimensional example) that are linearly independent and that satisfy the selection rule developed in Appendix A gives the following set of 6 vector pairs.

$\underline{v}(1,2)$ and $\underline{v}(1,3)$
$\underline{v}(1,2)$ and $\underline{v}(3,2)$
$\underline{v}(1,3)$ and $\underline{v}(2,3)$
$\underline{v}(2,1)$ and $\underline{v}(2,3)$
$\underline{v}(2,1)$ and $\underline{v}(3,1)$
$\underline{v}(3,1)$ and $\underline{v}(3,2)$

Thus, we can form at most 6 hyperplanes. Some of these may be redundant as will happen with this example since the tradeoffs between MOEs 1 and 2 are linear.

The following two non-redundant hyperplane vectors are obtained by solving the pair of equations obtained when the dot products of each pair of vectors with the hyperplane vectors are set equal to zero.

$$
\begin{aligned}
\underline{w}^1 &= (\ 1, & 0.6, & \quad 0.35) \\
\underline{w}^2 &= (\ 1, & 0.6, & \quad 1.43)
\end{aligned} \tag{19}
$$

One element of each hyperplane is arbitrary at this point, and we selected the first elements to be equal to unity. To determine whether these hyperplanes are both allowed, we compute their dot products with each of the tradeoff ratio vectors. These are all greater than or equal to zero. Thus, these two hyperplanes are allowed.

To determine whether all tradeoff ratio vectors are included, we check that at least one dot product of each vector with the allowed hyperplanes is zero. This is the case, and we conclude that our piecewise linear model consists of the above two hyperplanes. The corresponding unit hyperplane vectors are:

$$\begin{aligned} \underline{\alpha}^1 &= (0.82, \quad 0.493, \quad 0.287) \\ \underline{\alpha}^2 &= (0.542, \quad 0.325, \quad 0.775) \end{aligned} \tag{20}$$

E. <u>The Objective Function Formulation</u>

Given $\underline{\alpha}^1$, $\underline{\alpha}^2$, $\underline{\epsilon}$, and $\underline{a}$, we form the following objective function

$$g(\underline{x}) = \mathrm{Min}\left[\frac{\underline{\alpha}^1\cdot(\underline{x} - \underline{a})}{\underline{\alpha}^1\cdot\underline{\varepsilon}}, \frac{\underline{\alpha}^2\cdot(\underline{x} - \underline{a})}{\underline{\alpha}^2\cdot\underline{\varepsilon}}\right] \tag{21}$$

To test this function numerically consider two outcomes $\underline{x}^1$ and $\underline{x}^2$ given by

$$\begin{aligned} \underline{x}^1 &= (4,5,7.4) \\ \underline{x}^2 &= (4,6.67, 6) \end{aligned} \tag{22}$$

The evaluation of $g(\underline{x})$ for these two outcomes given

$$g(\underline{x}^1) = \mathrm{Min}\left[\frac{0.402}{0.925}, \frac{1.090}{0.947}\right] = 0.435$$

$$g(\underline{x}^2) = \mathrm{Min}\left[\frac{0.823}{0.925}, \frac{0.543}{0.967}\right] = 0.562 \qquad (23)$$

These results show that the piecewise linear model indicates a preference for $\underline{x}^2$. We note that the value of $g(\underline{x}^1)$ was determined by hyperplane $\underline{\alpha}^1$, whereas the value of $g(\underline{x}^2)$ was determined by hyperplane $\underline{\alpha}^2$. We also note the roles of the weighting functions $\underline{\alpha}^1 \cdot \underline{\epsilon}$ and $\underline{\alpha}^2 \cdot \underline{\epsilon}$. In this example they do not differ by very much, and therefore, the numerator terms determine which hyperplane is active in each case. If the value of $\underline{\epsilon}$ were such that $\underline{\alpha}^1 \cdot \underline{\epsilon}$ was equal to 0.5, and $\underline{\alpha}^2 \cdot \underline{\epsilon}$ was equal to 1.5, then only hyperplane $\underline{\alpha}^2$ would be active, and $\underline{x}^1$ would become the preferred outcome.

F. Tradeoff Assessments Revisited (Consistency Checking)

This completes the illustrative numerical example for the case where consistent tradeoff ratio evaluations are obtained from the DM. Let us now consider the case where the DM cannot agree with the inferred value of $\gamma_{32} = 0.42$ and estimates the value of $\gamma_{32} = 0.50$. If the DM has a high confidence in this value, then we deduce that the constraint (see Eq. A-22) presents the consistency problem. Therefore, either γ_{31} is too low, or γ_{21} is too high. If γ_{31} has lower confidence than γ_{21}, then Eq. A-22 tightly constrains γ_{31} and we deduce that γ_{31} must be raised to 0.833. If the DM agrees with this value, the new tradeoff ratio matrix becomes

$$\begin{matrix} 1 & .6 & .35 \\ 1.67 & 1 & .583 \\ .833 & .5 & 1 \end{matrix}$$

All of these values now pass the consistency tests and the procedures for determining the hyperplanes should yield two hyperplanes as before.

If on the other hand, the value of γ_{21} has less confidence, it can be decreased to say 1.4. However, this change results in modifying the tradeoff between x_1 and x_2 to a nonlinear case. If this is accepted by the DM, the new tradeoff ratio matrix becomes

$$\begin{matrix} 1 & .6 & .35 \\ 1.4 & 1 & .583 \\ .7 & .5 & 1 \end{matrix}$$

All these values are now consistent with convexity, but more than two hyperplanes will be required in our model.

The tradeoff ratio vectors for this case are:

$$\begin{aligned}
\underline{v}(1,2) &= (-0.6, \quad 1, \quad 0) \\
\underline{v}(1,3) &= (-0.35, \quad 0, \quad 1) \\
\underline{v}(2,1) &= (\ 1 \quad -1.4, \quad 0) \\
\underline{v}(2,3) &= (\ 0, \quad -0.583, \quad 1) \qquad (24) \\
\underline{v}(3,1) &= (\ 1, \quad 0, \quad -0.7) \\
\underline{v}(3,2) &= (\ 0, \quad 1, \quad -0.5)
\end{aligned}$$

The following four non-redundant hyperplane vectors are obtained.

$$\begin{aligned}
\underline{w}^1(1) &= (\ 1, \quad 0.6 \quad 0.35) \\
\underline{w}^2(1) &= (\ 1, \quad 0.6 \quad 1.25) \\
\underline{w}^3(1) &= (\ 1, \quad 0.714, \quad 0.416) \qquad (25) \\
\underline{w}^4(1) &= (\ 1, \quad 0.714, \quad 1.43)
\end{aligned}$$

These hyperplanes are all allowed and include all the tradeoff ratio vectors. Thus, our piecewise linear model now consists of four hyperplanes. The next step is to normalize these to unit vectors and proceed with the remaining procedures as before.

Appendix A

Given a set of tradeoff ratios obtained by a sequence of tradeoff assessments between pairs of MOEs, we are interested in selecting a set of hyperplanes that form a piecewise linear approximation to an "indifference" hypersurface.

For an n-dimensional problem, it is possible to form n(n -1)/2 pairs of MOEs. However, since we are considering the case where a tradeoff assessment yields different results, depending on the direction of change of the reference MOE, there are two possible tradeoff ratios for each tradeoff pair. Thus, we are dealing with n(n-1) distinct tradeoff ratios. Each of these tradeoff ratios can be formed into a vector with only two non-zero terms. The resulting vectors can then be considered as sample points from the indifference hypersurface. A piecewise linear approximation to this hypersurface can be obtained by first finding all the hyperplanes that each contain n-1 linearly independent vectors taken from the total set of n(n-1) vectors. The total number of possible hyperplanes is then given by

$$m = \frac{[n(n - 1)]!}{[n - 1]![n(n - 1) - (n - 1)]!} \qquad \text{(A-1)}$$

However, by our assumption of increasing marginal rate of substitution of MOEs we are only interested in those hyperplanes that form a convex piecewise linear hypersurface. This restriction reduces the number of hyperplanes required in the model. For example, a three-dimensional problem yields six tradeoff ratio vectors and the possibility of forming 15 hyperplanes. Applying the convexity restrictions reduces the required number of hyperplanes to six.

The purpose of the remainder of this appendix is to develop the procedures for forming and selecting the required hyperplanes, and to develop certain tradeoff ratio bounds that are useful in evaluating and inferring tradeoff ratio values.

A. Forming and Selecting Hyperplanes

The tradeoff ratio vectors discussed above need only be defined to within a scale factor to derive the hyperplanes. Thus, they are defined as

$$\underline{v}(i,j) = (0, \ldots, \underset{\substack{\text{i-th}\\ \downarrow}}{-\gamma_{ij}}, \ldots, \underset{\substack{\text{j-th}\\ \downarrow}}{1}, \ldots, 0) \qquad \text{(A-2)}$$

The term γ_{ij} is defined as the tradeoff ratio obtained by increasing MOE_j and decreasing MOE_i. It will always be greater than zero.

A hyperplane is also defined by a vector. In particular, a vector $\underline{w}$ represents a hyperplane containing a vector $\underline{v}$ if the dot product of the two vectors is zero. Thus, the vector $\underline{w}$ is also defined to within a scale factor. Proper scaling is achieved by normalizing $\underline{w}$ to a unit vector .

A set of n-1 tradeoff ratio vectors define a hyperplane vector $\underline{w}$ by the solution of the (n-1) dot product equations of the form

$$\underline{v}(i,j) \cdot \underline{w} = w_j - \gamma_{ij} w_i = 0 \qquad \text{(A-3)}$$

where $i \neq j$, $\underline{v}(i,j)$ is taken from the set of n-1 vectors, and w_k is arbitrarily set to 1. With $w_k = 1$ we designate the hyperplane vector $\underline{w}(k)$. Two particularly simple forms of $\underline{w}(k)$ result if either the set of tradeoff ratio vectors consist of $\underline{v}(k,i)$ or $\underline{v}$ (i,k) for i = 1 to n and $i \neq k$. The first case yields

$$\underline{w}(k) = (\gamma_{k1}, \gamma_{k2}, \ldots, \underset{\text{k-th}}{1}, \ldots, \gamma_{kn}) \tag{A-4}$$

The second yields

$$\underline{w}(k) = \left(\frac{1}{\gamma_{1k}}, \frac{1}{\gamma_{2k}}, \ldots, \underset{\text{k-th}}{1}, \ldots, \frac{1}{\gamma_{nk}}\right) \tag{A-5}$$

Other sets of n-1 vectors will yield $\underline{w}(k)$ terms with products of s in the numerators and/or denominators.

It will be shown that certain combinations of $\underline{v}$'s will result in hyperplanes that violate the convexity assumption. In particular, the vector pair $\underline{v}(i,j)$ and $\underline{v}(k,i)$, as well as the vector pair $\underline{v}(i,j)$ and $\underline{v}(j,k)$ lead to convexity violations for any value of k.

In addition, the vector pair $\underline{v}(k,i)$ and $\underline{v}(j,i)$, as well as the vector pair $\underline{v}(i,k)$ and $\underline{v}(i,j)$ may lead to convexity violations unless the values of the ratios γ_{ki}/γ_{ji} and γ_{ij}/γ_{ik}, respectively, fall within certain limits.

The convexity assumption can be translated into two conditions:

(1) For any given i and j, $\gamma_{ij} \leq 1/\gamma_{ji}$

(2) The dot product of each tradeoff ratio vector with any hyperplane vector used in the piecewise linear model, must be greater than or equal to zero. (It will equal zero if, and only if, the tradeoff ratio vector lies in the hyperplane under consideration.)

In addition to the convexity assumption, we have an inclusion condition that requires that each tradeoff ratio vector be contained in at least one hyperplane that does not violate the convexity condition. The violation of this inclusion condition means that the tradeoff assessment data in that vector is inconsistent with the convexity condition. If this occurs, two options are available. The first is to reassess the tradeoff ratio until it is consistent with convexity. The second approach is to simply consider it as a "bad" datum, discard it, and form a limited piecewise linear indifference model with the remaining "good" data.

Since we are interested in modelling possible nonlinearities, we consider only the case for which γ_{ij} is strictly less than $1/\gamma_{ji}$, and require that any tradeoff ratio vector that is not used to form a given hyperplane, has a positive non-zero dot product with that hyperplane vector.

To show that a hyperplane that includes vectors $\underline{v}(i,j)$ and $\underline{v}(k,i)$ violates the convexity assumption, we proceed by selecting two hyperplanes and show that if one violates the convexity condition, then the other does not, and vice versa. It may be true that both violate the convexity condition, but we are not concerned with that outcome at this point. We then show that if we choose to assume that one of these hyperplanes are allowed, then one particular choice leads to a violation of the inclusion condition.

Let $\underline{u}(k)$ be a hyperplane containing the vectors $\underline{v}(k,i)$ and $\underline{v}(i,j)$, and $\underline{w}(k)$ be a hyperplane containing $\underline{v}(k,i)$ and $\underline{v}(k,j)$. The evaluation of the i,j, and k terms of $\underline{u}(k)$ and $\underline{w}(k)$ results in

$$\underline{u}(k) = (\ldots,\ \overset{i\text{-th}}{\overset{\downarrow}{\gamma_{ki}}},\ \ldots,\ \overset{j\text{-th}}{\overset{\downarrow}{\gamma_{ki}\gamma_{ij}}},\ \ldots,\ \overset{k\text{-th}}{\overset{\downarrow}{1}}\ ,\ \ldots) \tag{A-6}$$

$$\underline{w}(k) = (\ldots,\ \overset{i\text{-th}}{\overset{\downarrow}{\gamma_{ki}}},\ \ldots,\ \overset{j\text{-th}}{\overset{\downarrow}{\gamma_{kj}}},\ \ldots,\ \overset{k\text{-th}}{\overset{\downarrow}{1}}\ ,\ \ldots) \tag{A-7}$$

Consider the dot product of $\underline{v}(k,j)$ with $\underline{u}(k)$, and the dot product of $\underline{v}(i,j)$ with $\underline{w}(k)$. this gives

$$\underline{v}(k,j)\cdot\underline{u}(k) = -\gamma_{kj} + \gamma_{ki}\gamma_{ij} \tag{A-8}$$

and

$$\underline{v}(i,j)\cdot\underline{w}(k) = \gamma_{kj} - \gamma_{ki}\gamma_{ij} \tag{A-9}$$

These two equations show that if one dot product is greater than zero, the other is less than zero. Thus, either $\underline{u}(k)$ or $\underline{w}(k)$, or possibly both, are not allowed.

Now assume that $\underline{u}(k)$ is allowed and determine the form of an arbitrary hyperplane $\underline{y}(k)$ containing $\underline{v}(k,j)$. If this form of hyperplane is not allowed, then $\underline{v}(k,j)$ would violate the inclusion condition. The form of $\underline{y}(k)$ is

$$\underline{y}(k) = (\ldots,\ \overset{j\text{-th}}{\overset{\downarrow}{\gamma_{kj}}},\ \ldots,\ \overset{k\text{-th}}{\overset{\downarrow}{1}}\ ,\ \ldots) \tag{A-10}$$

The dot products of the vectors $\underline{v}(k,i)$ and $\underline{v}(i,j)$ with $\underline{y}(k)$ give

$$\underline{v}(k,i)\cdot\underline{y}(k) = y_i - \gamma_{ki} \tag{A-11}$$

$$\underline{v}(i,j)\cdot\underline{y}(k) = \gamma_{kj} - \gamma_{ij}y_i \tag{A-12}$$

Combining these equations to eliminate y_i gives

$$\underline{v}(i,j)\cdot\underline{y}(k) = (\gamma_{kj} - \gamma_{ki}\gamma_{ij}) - \gamma_{ij}\,\underline{v}\,(k,i)\cdot\underline{y}\,(k) \qquad \text{(A-13)}$$

But, the first term on the right of the equality is $\underline{v}(i,j)$ $\underline{w}(k)$ which must be less than zero by our assumption that $\underline{w}(k)$ is not allowed. Thus, clearly if $\underline{v}(k,i)$ $\underline{y}(k)$ is greater than zero as it must be to allow $\underline{y}(k)$, then $\underline{v}(i,j)$ $\underline{y}(k)$ is less than zero. The conclusion is that the form of $\underline{y}(k)$ is not allowed and therefore there is no allowed hyperplane containing $\underline{v}(k,j)$. Thus, $\underline{v}(k,j)$ violates the inclusion condition under the assumption that $\underline{u}(k)$ is allowed.

The above result shows that whereas $\underline{w}(k)$ may or may not be allowed in order that the inclusion condition hold, $\underline{u}(k)$ cannot be allowed since the convexity condition would be violated.

In an analogous manner we can select $\underline{u}(k)$ as a hyperplane containing $\underline{v}(j,k)$ and $\underline{v}(i,j)$, and $\underline{w}(k)$ as a hyperplane containing $v(i,k)$ and $\underline{v}(j,k)$, and again show that $\underline{u}(k)$ is not allowed.

We finally conclude that vector pairs such as $\underline{v}(i,j)$ and $\underline{v}(k,i)$, as well as $\underline{v}(i,j)$ and $\underline{v}(j,k)$ are not allowed when all tradeoff assessments are consistent with convexity and satisfy the inclusion condition.

However, we also note that if a tradeoff ratio vector turns out to be inconsistent due to the inclusion condition and is then excluded from the set of possible tradeoff ratio vectors, the vector pairs and certain hyperplanes that they are contained in would no longer be excluded. Thus, these vector pairs provide a means to check for tradeoff assessment consistency with the convexity assumption, and a means to form a piecewise linear indifference model when and if inconsistent tradeoff assessments are excluded.

The above results provide restrictions on the number of allowed hyperplanes that result from convexity assumption requirements. In addition, certain hyperplanes that may be allowed by the convexity assumption will not be allowed when specific numerical values of tradeoff ratios are used. In particular, the ratio of certain tradeoff ratios must fall within certain bounds.

B. Tradeoff Ratio Bounds and Inferences

To derive limits on the allowed ratio of γ_{ki}/γ_{ji} for a hyperplane containing $\underline{v}(k,i)$ and $\underline{v}(j,i)$ to be allowed we proceed by first assuming that the hyperplane, call it $\underline{u}(k)$, is allowed. By the previous proof, we observe that $\underline{v}(k,j)$ and $\underline{v}(j,k)$ cannot then be contained in $\underline{u}(k)$. The form of $\underline{u}(k)$ will be

$$\underline{u}(k) = (\ldots, \underset{\substack{\text{i-th}\\ \downarrow}}{\gamma_{ki}}, \ldots, \underset{\substack{\text{j-th}\\ \downarrow}}{\gamma_{ki}/\gamma_{ji}}, \ldots, \underset{\substack{\text{k-th}\\ \downarrow}}{1}, \ldots) \tag{A-14}$$

The dot product of $\underline{v}(k,i)$ and $\underline{v}(j,k)$ with $\underline{u}(k)$ must be greater than or equal to zero according to the convexity condition. Thus,

$$\underline{v}(k,j)\cdot\underline{u}(k) = -\gamma_{kj} + \gamma_{ki}/\gamma_{ji} \geq 0 \tag{A-15}$$

and

$$\underline{v}(j,k)\cdot\underline{u}(k) = -\gamma_{jk}\gamma_{ki}/\gamma_{ji} + 1 \geq 0 \tag{A-16}$$

These two equations yield

$$\gamma_{kj} \leq \gamma_{ki}/\gamma_{ji} \leq 1/\gamma_{jk} \tag{A-17}$$

Thus, unless the ratio γ_{ki}/γ_{ji} is within the above bounds, $\underline{u}(k)$ will not be allowed.

In an exactly analogous manner, we can show that a hyperplane containing $\underline{v}(i,k)$ and $\underline{v}(i,j)$ is allowed only if the limits on the ratio γ_{ij}/γ_{ik} are given by

$$\gamma_{kj} \leq \gamma_{ij}/\gamma_{ik} \leq 1/\gamma_{jk} \tag{A-18}$$

Given a consistent set of tradeoff ratios, we have shown that certain combinations of vector pairs when used to define a hyperplane will always result in disallowed hyperplanes regardless of numerical values. In addition, specific numerical values of the tradeoff ratios may lead to the disallowal of additional hyperplanes.

These results also allow us to establish bounds on any given tradeoff ratio. Consider the arbitrary tradeoff ratio γ_{kj}. The convexity and inclusion assumptions require that Eqs. A-17 and A-18 be both simultaneously satisfied. Also since Eq. A-17 holds for arbitrary i,j, and k, we can reverse the roles of i and j and write,

$$\gamma_{ki} \leq \gamma_{kj}/\gamma_{ij} \tag{A-19}$$

Eq. A-19 provides a lower bound on γ_{kj}. In addition, we can write Eq. A-17 with the roles of k and i reversed to give,

$$\gamma_{ik}/\gamma_{jk} \leq 1/\gamma_{ji} \tag{A-20}$$

From Eq. A-20 we deduce that

$$1/\gamma_{jk} \leq 1/(\gamma_{ji}\gamma_{ik}) \tag{A-21}$$

Combining all these results gives,

$$\gamma_{ki}\gamma_{ij} \leq \gamma_{kj} \leq \mathrm{Min}\left[\frac{\gamma_{ij}}{\gamma_{ik}}, \frac{\gamma_{ki}}{\gamma_{ji}}, \frac{1}{\gamma_{ji}\gamma_{ik}}, \frac{1}{\gamma_{jk}}\right] \tag{A-22}$$

Further, note that if the pairwise tradeoff between i and j is linear (i.e. $\gamma_{ij} = 1/\gamma_{ji}$), then the lower and upper bounds on γ_{kj} are equal, and we must have that

$$\gamma_{kj} = \gamma_{ki}/\gamma_{ji} \tag{A-23}$$

Similarly if the pairwise tradeoff between k and i is linear (i.e. $\gamma_{ki} = 1/\gamma_{ik}$), then

$$\gamma_{kj} = \gamma_{ki}\gamma_{ij} \tag{A-24}$$

It is interesting to interpret each of the bounds given in Eq. A-22. They are the inferred value of γ_{kj} under the assumption that the indifference surface in the 3-dimensional subspace is linear, and the tradeoff ratio vectors corresponding to the terms in the bounds lie in the subspace plane for the indifference surface. For example, if the plane containing $\underline{v}$ (k,i) and $\underline{v}$ (j,i) also contains $\underline{v}$ (k,j), then the value of γ_{kj} be inferred to be equal to γ_{ki}/γ_{ji}. The convexity and inclusion conditions establish the inequalities when the subspace surface is not a plane.

Eq. A-22 provides a very useful consistency check on the tradeoff ratios without having to determine sets of hyperplanes, and when linear pairwise tradeoffs occur, Eq. A-23, or A-24 allows us to infer the values of certain tradeoff ratios.

Appendix B

The linear objective function derived for the Basic RA Method can be written as

$$g(\underline{x}) = \sum_{i=1}^{n} \gamma_{ni} x_i \qquad \text{(B-1)}$$

where γ_{ni} is the tradeoff ratio between the n-th and i-th MOEs. These are determined from the decision maker's responses to the tradeoff assessment procedures. The γs define a hyperplane in n-dimensional space that is tangent to an indifference hypersurface at a point in the neighborhood of those used in the tradeoff assessment procedures.

The Extended RA Method assumes that for certain sets of MOE pairs, the linear approximation to the indifference hypersurface may be a poor representation. In particular, a piecewise linear approximation may be more appropriate as illustrated in Figure B-1 for the two-dimensional case. Here we see that there is a certain "most preferred proportion" between MOEs x_1 and x_2. The direction defined by this proportion represents the direction of steepest ascent along the true preference function. The linear objective function approximation to the left will be called $g_1(x)$ and the other, $g_2(\underline{x})$. Applying the Basic RA optimization procedure separately to $g_1(\underline{x})$ and $g_2(\underline{x})$ will yield as the most preferred alternatives, $\underline{d}$ and $\underline{b}$, respectively. By construction, we see that $\underline{c}$ should indeed be the most preferred alternative. Thus, a new nonlinear functional is required that combines $g_1(\underline{x})$, $g_2(\underline{x})$, and the most preferred proportion information, and yields $\underline{c}$ as the most preferred alternative.

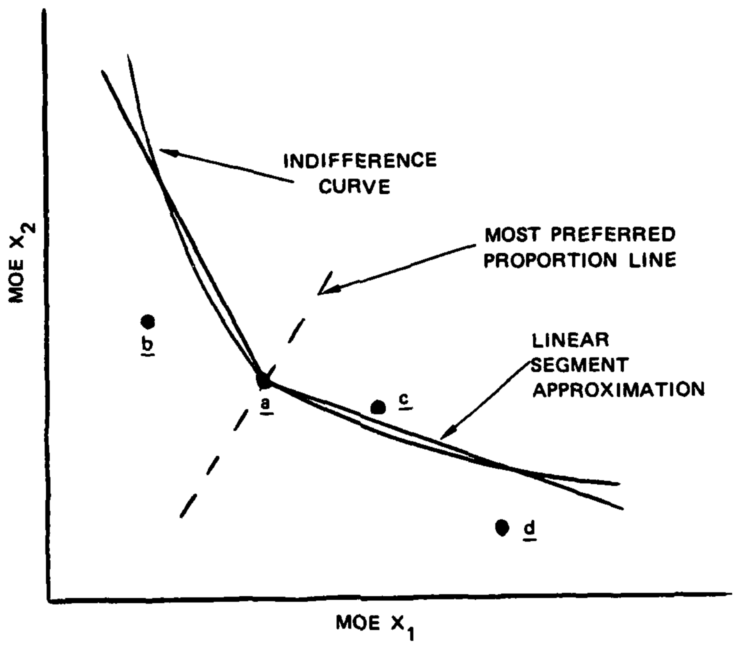


FIGURE B-1 PIECEWISE LINEAR APPROXIMATION

To facilitate the development of the appropriate piecewise linear objective function, the vector of s can be normalized to a unit vector, $\underline{\alpha}$. The elements of $\underline{\alpha}$ are then the direction cosines of the normal to the hyperplane. Let $\underline{\alpha}$ represent the direction cosines corresponding to the γs for $g_1(\underline{x})$, and $\underline{\beta}$ represent those for $g_2(x)$. We can then define

$$g_1(\underline{x}) = \underline{\alpha} \cdot \underline{x} = \sum_{i=1}^{n} \alpha_i x_i \qquad \text{(B-2)}$$

and

$$g_2(\underline{x}) = \underline{\beta} \cdot \underline{x} = \sum_{i=1}^{n} \beta_i x_i \qquad \text{(B-3)}$$

Note that the values for these functions now give the distance of each hyperplane from the origin. These representations are illustrated in Figure B-2.

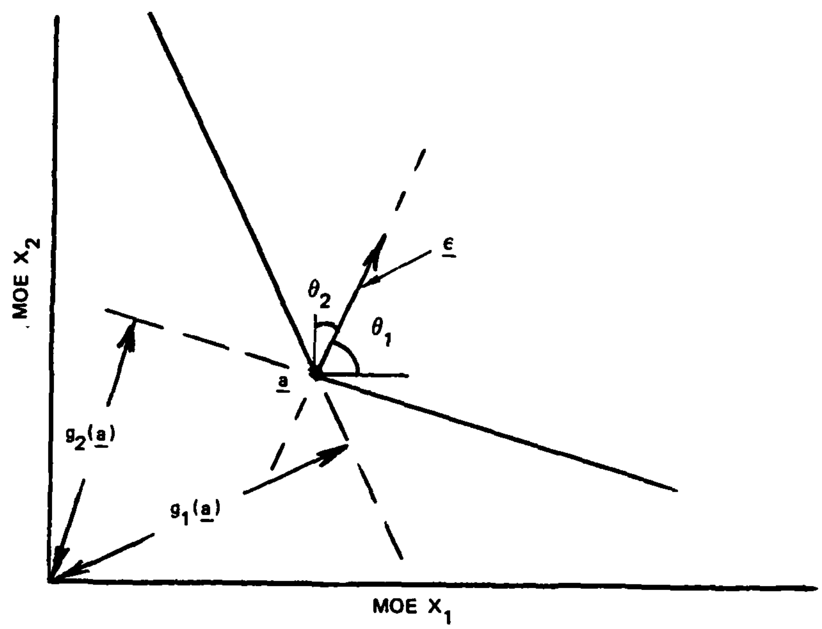


Figure B-2 LINEAR FUNCTIONAL REPRESENTATIONS

Assume now that the most preferred proportion has a direction cosine vector, $\underline{\epsilon}$. The elements of $\underline{\epsilon}$ are cos θ_1 and cos θ_2, respectively, where the θs are defined in Figure B-2. Now consider the function

$$g(\underline{x}) = \mathrm{Min}\left[\frac{g_1(\underline{x} - \underline{a})}{g_1(\underline{\epsilon})}, \frac{g_2(\underline{x} - \underline{a})}{g_2(\underline{\epsilon})}\right] \qquad \text{(B-4)}$$

The value of this function is zero for any point that lies on the solid line to the left of $\underline{a}$ and going through $\underline{a}$, or on the solid line to the right of $\underline{a}$ and going through $\underline{a}$. For any point above these two lines, the value will be a function of g_1 if the point

is to the left of the $\underline{\epsilon}$ vector, or a function of g_2 if to the right of $\underline{\epsilon}$. In Eq. B-4, the values $g_1(\underline{\epsilon})$ and $g_2(\underline{\epsilon})$ serve as weights between the values of g_1 and g_2.

This function readily generalizes to n-dimensions by simply adding terms to the minimization function list for each hyperplane used in the piecewise linear model. The determination of these hyperplanes is discussed Appendix A. The objective function then becomes

$$g(\underline{x}) = \underset{i=1,h}{\text{Min}} \left[\frac{g_i(\underline{x} - \underline{a})}{g_i(\underline{\epsilon})} \right] \tag{B-5}$$

where h is the number of hyperplanes.

For a specific case, the value of $g(\underline{x})$ will be determined by one of the h hyperplanes. The particular hyperplane that determines $g(\underline{x})$ is controlled by the set of weighting functions $g_i(\underline{\epsilon})$.

The optimization problem for finding the most preferred alternative then becomes:

Find $\underline{x}^*$ such that

$$g(\underline{x}^*) = \underset{\underline{x}}{\text{Max}} \left\{ \underset{i=1,h}{\text{Min}} \left[\frac{g_i(\underline{x} - \underline{a})}{g_i(\underline{\epsilon})} \right] \right\} \tag{B-6}$$

Appendix C

The evaluation of the components of the most preferred proportion vector can be carried out employing additional tradeoff assessment information from the decision maker. Figure C-1 illustrates the problem for the 2-dimensional case with MOEs arbitrarily labelled x_n and x_i, or equivalently, the projection of the n-dimensional case onto the (x_i, x_n) plane. The tangents of the angles θ_1 and θ_2 are defined by tan γ_{in} and tan γ_{ni}, respectively. These γs are values obtained from the matrix of indifference tradeoff ratios. The vector $\underline{\epsilon}$ is the unit vector that defines the direction for the most preferred proportion of x_n and x_i.

Consider the following tradeoff assessment. A hypothetical outcome $\underline{b}$ is constructed from $\underline{a}$ by increasing the value of x_n to some convenient level. For example, the change in x_n might be calculated assuming an incremental change in budget that is allocated to the improvement of x_n only. The point $\underline{b}$ will lie on some new indifference curve. The most preferred proportion assumption implies that all hyperplanes intersect at the locus of points along the most preferred proportion direction as indicated in Fig. C-1 by the dashed line.

Given $\underline{a}$ and $\underline{b}$, a positive tradeoff assessment can now be performed by asking the decision maker how much x_i would have to be increased to compensate for a decrease in x_n from the level at $\underline{b}$ to the level at $\underline{a}$. The decision maker's response allows us to construct a new outcome $\underline{c}$ that lies at the same indifference level as $\underline{b}$. The outcome $\underline{b}$ is constructed by increasing the x_i value at $\underline{a}$ by the amount of the decision maker's response. Given $\underline{b}$ and $\underline{c}$ the graphical solution for $\underline{\epsilon}$ is obtained by constructing the two lines forming the indifference curve containing $\underline{b}$ and $\underline{c}$ and determining their intersection point. The vector $\underline{\epsilon}$ must then be along the direction from this intersection and point $\underline{a}$.

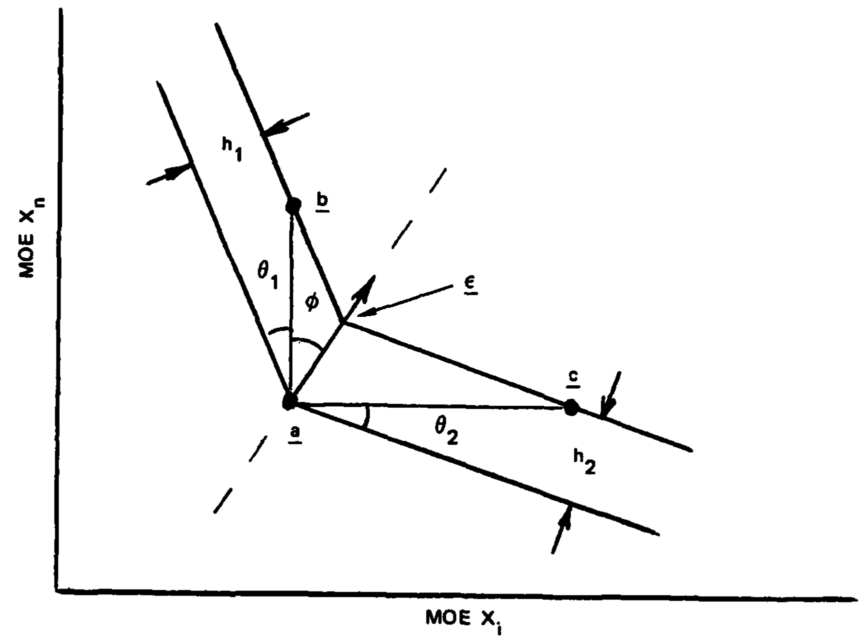


FIGURE C-1 MOST PREFERRED PROPORTION PARAMETERS

The mathematical solution of this problem proceeds as follows. Let ϕ be the angle between $\underline{\epsilon}$ and the x_n axis. The distances h_1 and h_2 in Figure C-1 are given by

$$h_1 = (b_n - a_n) \sin \Theta_1 \tag{C-1}$$

and

$$h_2 = (c_i - a_i) \sin \Theta_2 \tag{C-2}$$

where

$$\tan \Theta_1 = \gamma_{in} \tag{C-3}$$

and

$$\tan \Theta_2 = \gamma_{ni} \tag{C-4}$$

The relationship between h_1 And h_2 is given by

$$\frac{h_1}{\sin(\Theta_1 + \phi)} = \frac{h_2}{\sin(\Theta_2 + 90 - \phi)} \tag{C-5}$$

Solving Eq. (C-5) for ϕ yields

$$\tan \phi = \frac{h_2 \sin \Theta_1 - h_1 \sin \Theta_2}{h_1 \sin \Theta_2 - h_2 \cos \Theta_1} \tag{C-6}$$

The ratio of $(b_n - a_n)$ to $(c_i - a_i)$ is an equivalent negative tradeoff ratio between outcomes $\underline{b}$ and $\underline{c}$, and can be defined as γ'_{ni}

$$\gamma'_{ni} = \frac{b_n - a_n}{c_i - a_i} \tag{C-7}$$

Using Eqs. C-1 through C-4, and Eq. C-7, the expression for tan ϕ becomes

$$\tan \phi = \frac{1 - \gamma_{ni}'/\gamma_{ni}}{\gamma_{ni}' - 1/\gamma_{in}} \qquad \text{(C-8)}$$

Eq. C-8 allows us to determine ϕ and the elements of $\underline{\epsilon}$ for the 2-dimensional case by setting i=1 and n=2, and noting that is defined as

$$\underline{\varepsilon} = (\sin \phi, \cos \phi) \qquad \text{(C-9)}$$

This is equivalent to

$$\underline{\varepsilon} = \frac{1}{\sqrt{1 + \tan^2 \phi}} (\tan \phi, 1) \qquad \text{(C-10)}$$

In the higher dimensional cases, we can denote ϕ as ϕ_{ni} and compute tan ϕ_{ni} for i=1 through n-1 from

$$\tan \phi_{ni} = \frac{1 - \gamma_{ni}'/\gamma_{ni}}{\gamma_{ni}' - 1/\gamma_{in}} \qquad \text{(C-11)}$$

The vector $\underline{\epsilon}$ is then given by

$$\underline{\varepsilon} = \frac{(\tan \phi_{ni}, \tan \phi_{n2}, \ldots, 1)}{\left[1 + \sum_{i=1}^{n} (\tan \phi_{ni})^2\right]^{1/2}} \qquad \text{(C-12)}$$

Eq. C-8 will be indeterminate when there is a linear pairwise tradeoff between MOEs n and i since in that case $\gamma_{ni}=1/\gamma_{in}$ and $\acute{\gamma}_{ni}$ will be equal to γ_{ni}. When this occurs, Eq. C-11 cannot be used since we will not be able to determine the value of tan ϕ_{ni}. In that case, we can observe that the vector $\underline{\epsilon}$ represents a hyperplane and a procedure similar to that described in Appendix A can be employed to find $\underline{\epsilon}$. To employ this procedure, we first define an equivalent tradeoff ratio vector for any pair of MOE's that has only two non-zero terms. This has the form,

$$\underline{v}(k,j) = (0, \ldots, \overset{\substack{\text{k-th} \\ \downarrow}}{-\tan \phi_{kj}}, \ldots, 1, \ldots, 0) \tag{C-13}$$

Given n-1 linearly independent vectors that span the MOE space, we can compute a vector $\underline{w}$ such that its dot product with each of the n-1 vectors is zero. The hyperplane vector $\underline{w}$ is obtained by solving the n-1 dot product equations

$$\underline{v}^{i} \cdot \underline{w} = 0 \text{ for } i=1, n-1 \tag{C-14}$$

for n-1 elements of $\underline{w}$ in terms of one remaining arbitrary element that can be set equal to unity. The vector $\underline{\epsilon}$ is then given by

$$\underline{\varepsilon} = \frac{\underline{w}}{|\underline{w}|} \tag{C-15}$$

It will always be possible to form the required n-1 vectors unless all pairwise tradeoffs are linear. In this case, the vector $\underline{\epsilon}$ can be set equal to the linear indifference model hyperplane vector $\underline{\alpha}$.